A mod-\(\ell\) vanishing theorem of Beilinson-Soulé type

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Abstract

Let \(L\) be a field containing the algebraic closure of its prime field and \(X\) an equidimensional quasiprojective scheme over \(L\). We prove that \(CH^m(X, n; \mathbb{Z}/\ell) = 0\) when \(n > 2m \geq 0\) and \(\ell \neq 0\); this was known previously for \(m \geq \dim X\) in the case that \(X\) was smooth over an algebraically closed field. This “mod-\(\ell\)” version of the Beilinson-Soulé conjecture implies the equivalence of the rational and integral versions of the conjecture for varieties over fields of this type, and can be used to prove the vanishing of the (integral) groups \(CH^m(X, n)\) for \((n > 2m)\) in certain cases.

1 Introduction

Let \(k\) be a field and \(X\) a smooth variety over \(k\). The Beilinson-Soulé conjecture asserts that the motivic cohomology groups \(H^i(X, \mathbb{Z}(n))\) vanish for \(i < 0\) and any \(n\); by the work of Suslin and Voevodsky (see [V2]), this is equivalent to requiring that Bloch’s higher Chow groups \(CH^m(X, n)\) vanish when \(n > 2m\). It follows from the calculations in [B1], Section 7 that the conjecture holds for \(m = 0, 1\). While the conjecture itself seems very difficult, an analogous conjecture with finite coefficients seems somewhat more tractable: the work of Suslin [Su2], Geisser [G], and Geisser-Levine [GL] implies that when \(k\) is algebraically closed, \(X\) is smooth over \(k\), \(\ell \neq 0\), \(m \geq \dim X\), and \(n > 2m\), then \(CH^m(X, n; \mathbb{Z}/\ell) = H^{2m-n}_{\et}(X, \mathbb{Z}/\ell(n)) = 0\).

We extend this latter vanishing result, removing the hypotheses of smoothness and the range of \(m\), and relaxing the hypothesis on \(k\) – requiring only that \(k\) contain an algebraically closed field. The main result is the following:

Theorem.

Let \(L\) be a field containing the algebraic closure of its prime field, and \(X\) an equidimensional quasiprojective scheme over \(L\). Let \(\ell, m, n\) be integers such that \(\ell \neq 0\),
$m \geq 0$, and $n > 2m$. Then $CH^m(X, n; \mathbb{Z}/\ell) = 0$. Moreover, if $m \geq 1$ then $CH^m(X, n)$ is uniquely divisible for $n \geq 2m$.

As noted by Voevodsky in the introduction to [V1], this result is a consequence of the Bloch-Kato Conjecture. Recently, a proof of the Bloch-Kato conjecture has been announced by Voevodsky and Rost. Our proof, however, is much more elementary; it involves a generalization of Suslin’s rigidity theorem [Su1] due to Panin and Yagunov [PY], and employs an argument inspired by a result of Coombes [C]. The above theorem implies, among other things, that $CH^m(X, n)$ is uniquely divisible when $m \geq 1$ and $n \geq 2m$. Indeed, in this situation it suffices to know $CH^m(X, n) \otimes \mathbb{Q} = 0$ to deduce $CH^m(X, n) = 0$. Soulé [So] has proved that this is the case when $X$ belongs to a special class of varieties defined over the algebraic closure of a finite field; this class includes products of up to three curves, abelian surfaces, abelian threefolds, and unirational varieties of dimension $\leq 3$ and degree $\geq 3$.

Throughout this paper, we use the notation $nA$ for the $n$-torsion subgroup of an abelian group and $A/n$ for its $n$-cotorsion. If $R$ is a ring of geometric type, we often write $CH^*(R, \cdot)$ instead of $CH^*(\text{Spec } R, \cdot)$, and similarly for $K$-theory. If $X$ is an integral scheme over a field $L$, we write $L(X)$ for its field of rational functions, i.e. the local ring at the generic point of $X$. If $X$ is a scheme over $A$ and $B$ is an $A$-algebra, we write $X_B$ for the base extension $X \times_{\text{Spec } A} \text{Spec } B$. If $k \subseteq L$ is an extension of fields and $X$ is a $k$-scheme, we write $\iota_{L/k}(X) : X_L \to X$ for the morphism of schemes obtained by base extension.

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2 Rigidity

In [Su1] Suslin proves the Quillen-Lichtenbaum conjecture by means of a “rigidity” theorem for $K'$-theory with coefficients. In particular, it follows from his work that if $k \subseteq \Omega$ is an extension of algebraically closed fields and $X$ is a variety over $k$, then for $\ell$ relatively prime to the characteristic of $k$, the base change map $K'(X; \mathbb{Z}/\ell) \to K'(X_\Omega; \mathbb{Z}/\ell)$ is an isomorphism. Noting that Suslin uses various formal properties of the $K'$-theory functor in his argument (but nothing intrinsic to the definition of $K'$-theory), Panin and Yagunov [PY] have axiomatized Suslin’s proof to extend the scope of his result. In particular, they prove that functors with certain special formal properties satisfy an analogue of rigidity. The methods are closely related to those employed by Suslin and Voevodsky in Section 4 of [SV]; see also [MVW] 7.20.
Theorem 2.1. (Panin-Yagunov, [PY], Theorem 1.14) Let \( k \subseteq \Omega \) be an extension of algebraically closed fields, and \( Sm/k \) the category of smooth algebraic varieties over \( k \).
Suppose \( F: Sm/k \to Ab \) is a contravariant, homotopy invariant functor with weak transfers (cf. [PY], Definition 1.8) for the class of finite projective morphisms. Let \( n \geq 2 \) be an integer relatively prime to \( \text{char} \ k \) such that for every \( Y \), \( nF(Y) = 0 \). Then the natural map \( \pi: \text{Spec} \ \Omega \to \text{Spec} \ k \) induces an isomorphism \( \pi^*: F(\text{Spec} \ k) \xrightarrow{\cong} F(\text{Spec} \ \Omega) \).

Using well-known properties of higher Chow groups, it is a matter of routine to check that for any quasiprojective scheme \( X \) over \( k \) and any integers \( \ell, m, n \), \( \ell \neq 0 \), the functor \( F: Sm/k \to Ab \) defined by \( F(Y) = \text{CH}^m(X \times_k Y, n; \mathbb{Z}/\ell) \) is indeed a contravariant, homotopy invariant functor with weak transfers for the class of finite projective morphisms.

We therefore deduce the following:

Theorem 2.2. Let \( k \subseteq \Omega \) be an extension of algebraically closed fields and \( X \) a smooth projective variety over \( k \). Let \( \ell \neq 0 \) be any integer prime to \( \text{char} \ k \). Then there is a natural map inducing an isomorphism \( \text{CH}^*(X, \cdot; \mathbb{Z}/\ell) \to \text{CH}^*(X_{\Omega}, \cdot; \mathbb{Z}/\ell) \).

Since the localization sequence will be used repeatedly in the sequel, we state it here for reference:

Lemma 2.3. (Bloch, [B2])
Let \( k \) be a field, \( X \) an equidimensional quasiprojective scheme over \( k \). Let \( Z \subseteq X \) be a closed subscheme of codimension \( c \), and let \( U = X - Z \). Let \( A \) be either \( \mathbb{Z} \) or \( \mathbb{Z}/\ell \mathbb{Z} \) for some \( \ell \neq 0 \). Then for all \( m, n \in \mathbb{Z} \), there is a long exact sequence:

\[
\ldots \to \text{CH}^{m-c}(Z, n; A) \to \text{CH}^m(X, n; A) \to \text{CH}^m(U, n; A) \to \text{CH}^{m-c}(Z, n-1; A) \to \ldots
\]

We also note the following result for future reference:

Lemma 2.4. Let \( X \) be an integral equidimensional algebraic scheme over \( k \) and \( \eta \) the generic point of \( X \). The directed system consisting of neighborhoods \( U \) of \( \eta \) such that \( Z = X - U \) is equidimensional of codimension 1 in \( X \) is cofinal in the directed system of all neighborhoods of \( \eta \).

The following later result of Suslin (in characteristic 0) and Geisser (in arbitrary characteristic) will be crucial:
Theorem 2.5. ([Su2], Corollary 4.3 and [G], Corollary 3.6) Let $F$ be an algebraically closed field and $X$ a smooth equidimensional quasiprojective scheme over $F$. Then for any $i \geq \dim X$ and $\ell \neq 0$ relatively prime to $\text{char } F$, $CH^i(X, n; \mathbb{Z}/\ell) = H^{2i-n}_{\text{et}}(X, \mathbb{Z}/\ell(i))$.

In positive characteristic, the work of Geisser and Levine enables us to take care of the remaining case:

Theorem 2.6. ([GL], Theorem 1.1) Let $F$ be any field of characteristic $p > 0$. Then for $n \neq i$ and any $r \geq 1$, $CH^i(F, n; \mathbb{Z}/p^r) = 0$.

Corollary 2.7. Let $F$ be any field of characteristic $p > 0$ and $X$ any equidimensional quasiprojective scheme over $F$. Then for $n > 2i$ and any $r \geq 1$, $CH^i(X, n; \mathbb{Z}/p^r) = 0$.

Proof.
Let $\ell = p^r$. The proof proceeds by induction on $d = \dim X$, the case $d = 0$ following from Theorem 2.6. Suppose now $d > 0$; by working one component at a time, we may assume that $X$ is integral. Let $U$ be a neighborhood of the generic point of $X$ such that $Z = X - U$ has codimension 1. Since $n > 2i$, certainly $n - 1 > 2(i-1)$, so by induction $CH^{i-1}(Z, n; \mathbb{Z}/\ell) = CH^{i-1}(Z, n-1; \mathbb{Z}/\ell) = 0$. By localization (Lemma 2.3), $CH^i(X, n; \mathbb{Z}/\ell) = CH^i(U, n; \mathbb{Z}/\ell)$. By taking a direct limit over the system of all such neighborhoods $U$ of the generic point of $X$, we deduce from Lemma 2.4 that $CH^i(X, n; \mathbb{Z}/\ell) \cong CH^i(k(X), n; \mathbb{Z}/\ell)$. Now $n > 2i$, so by Theorem 2.6, $CH^i(k(X), n; \mathbb{Z}/\ell) = 0$, which completes the inductive step.

3 The Main Theorem

Although ultimately we will consider varieties over fields containing an algebraically closed field, the crucial part of the argument involves varieties over algebraically closed fields, as made precise in the following theorem.

Theorem 3.1. Let $k$ be an algebraically closed field, $X$ an equidimensional quasiprojective scheme over $k$ with structure map $\sigma : X \to \text{Spec } k$. Let $\ell$, $m$, $n$ be integers such that $\ell \neq 0$, $m \geq 0$, and $n > 2m$. Then $\sigma^* : CH^m(k, n; \mathbb{Z}/\ell) \to CH^m(X, n; \mathbb{Z}/\ell)$ is an isomorphism; so in particular, $CH^m(X, n; \mathbb{Z}/\ell) \cong CH^m(k, n; \mathbb{Z}/\ell) = 0$. If $X$ is irreducible, then $CH^m(k(X), n; \mathbb{Z}/\ell) = 0$.

By definition, the higher Chow groups of $X_{\text{red}}$ are isomorphic to those of $X$, so in the rest of this article we assume without loss of generality that $X$ is reduced. Moreover, by working one component at a time, the following is clear:
Lemma 3.2. If the conclusion of Theorem 3.1 holds when $X$ is irreducible, then it holds for arbitrary $X$.

The next Lemma, although somewhat obvious, will be important in jumpstarting an inductive proof.

Lemma 3.3. Let $k$ be any field, $X$ a quasiprojective scheme over $k$, and $\ell \neq 0$ any integer. Then $\text{CH}^0(X, n; \mathbb{Z}/\ell) = 0$ for any $n \geq 1$.

Lemma 3.4. If Theorem 3.1 holds when $X$ is smooth, then it holds when $X$ is singular.

Proof. Let $X$ be a singular variety; by Lemma 3.2 we may assume that $X$ is integral. The proof now proceeds by induction on $d = \dim X$. The case $d = 0$ is vacuous, so suppose $d > 0$ and let $U \subseteq X$ be a nonempty smooth open subset. By shrinking $U$ if necessary, we may assume without loss of generality that $Z = X - U$ has codimension 1 in $X$. By induction, $\text{CH}^{m-1}(Z, n; \mathbb{Z}/\ell) = 0$ and $\text{CH}^m(U, n; \mathbb{Z}/\ell) = 0$ because $U$ is smooth. Thus, by Lemma 2.3 $\text{CH}^m(X, n; \mathbb{Z}/\ell) = 0$.

Proof of Theorem 3.1

By Lemmas 3.2 and 3.4 we may assume that $X$ is integral and smooth. If $\dim X = 0$; that is, $X = \text{Spec } k$, then the result follows immediately from Theorem 2.5. We assume henceforth that $\dim X > 0$. Note that by Corollary 2.7, we may also assume without loss of generality that $\ell$ is relatively prime to $\text{char } k$.

We prove the theorem by induction on $m$. The case $m = 0$ is the content of Lemma 3.3. Now suppose the theorem is true for $m - 1$. Let $U \subseteq X$ be any nonempty, proper open subset; as in the proof of Lemma 3.4 we may assume that $Z = X - U$ has codimension 1 in $X$. By induction, $\text{CH}^{m-1}(Z, n; \mathbb{Z}/\ell) = 0$ and $\text{CH}^m(U, n; \mathbb{Z}/\ell) = 0$; hence by Lemma 2.3, $\text{CH}^m(X, n; \mathbb{Z}/\ell) \cong \text{CH}^m(U, n; \mathbb{Z}/\ell)$. By Lemma 2.4, $\text{CH}^m(X, n; \mathbb{Z}/\ell) \cong \text{CH}^m(k(X), n; \mathbb{Z}/\ell)$; let $j_X : \text{CH}^m(X, n; \mathbb{Z}/\ell) \to \text{CH}^m(k(X), n; \mathbb{Z}/\ell)$ be the map that gives this isomorphism.

Now let $\Omega$ be an algebraic closure of $k(X)$, and consider the following commutative diagram, in which (to simplify notation), we write $\alpha = \iota_{k(X)/k}(\text{Spec } k)^*$, $\beta = \iota_{k(X)/k}(\text{Spec } k(X))^*$, $\gamma = \iota_{\Omega/k(X)/k(\text{Spec } \Omega)^*$, $\delta = \iota_{\Omega/k(X)}$. 

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$CH^m(k, n; \mathbb{Z}/\ell)$

$\xymatrix{CH^m(k, n; \mathbb{Z}/\ell) \ar[r]^-{j_X^{-1}} \ar[d]^-{\alpha} & CH^m(X, n; \mathbb{Z}/\ell) \ar[d]^-{\beta} \\
CH^m(\Omega, n; \mathbb{Z}/\ell) \ar[r]^-{j_{\Omega}^{-1}} \ar[d]^-{\gamma} & CH^m(\Omega(X), n; \mathbb{Z}/\ell) \ar[d]^-{\delta} }

By Theorem 2.2, the maps $\beta \circ \alpha$ is an isomorphism, so $\alpha$ is injective and $\beta$ is surjective. Again by Theorem 2.2, $\delta$ is an isomorphism, so by commutativity, $\delta \circ j^{-1}_{X_0} = j^{-1}_X \circ \gamma \circ \beta$ is an isomorphism, and hence $\beta$ is injective. Thus, $\beta$ is an isomorphism, so $\alpha$ is an isomorphism, too. This implies that $\sigma^* = j^{-1}_X \circ \alpha : CH^m(k, n; \mathbb{Z}/\ell) \to CH^m(X, n; \mathbb{Z}/\ell)$ is an isomorphism. By Theorem 2.5, $CH^m(k, n; \mathbb{Z}/\ell) \cong H^0_{\text{ét}}(k, \mathbb{Z}/\ell(m)) = 0$ because $n > 2m$. This completes the inductive step.

**Corollary 3.5.** With notation as in Theorem 3.1, the group $CH^m(X, n)$ is torsion-free when $n = 2m$ and uniquely divisible when $n > 2m$. When $m \geq 1$, $CH^m(X, 2m)$ is uniquely divisible.

**Proof.**

From the universal coefficients exact sequence

$$0 \to CH^m(X, n)/\ell \to CH^m(X, n; \mathbb{Z}/\ell) \to CH^m(X, n - 1) \to 0$$

the only assertion that requires proof is the divisibility of $CH^m(X, 2m)$ when $m \geq 1$. In the following, we follow closely the arguments of [PW], Section 1 – the only difference being that we need to modify the argument slightly in positive characteristic.

Now suppose $\ell$ is prime. If $\ell = \text{char } k > 0$, then by Theorem 2.6, $CH^m(X, 2m; \mathbb{Z}/\ell) = 0$, so $CH^m(X, 2m)/\ell = 0$. Suppose $\ell \neq \text{char } k$. Let $U \subseteq X$ be open with $Z = X - U$ of codimension 1. By Theorem 3.1, $CH^{m-1}(Z, 2m; \mathbb{Z}/\ell) = CH^{m-1}(Z, 2m-1; \mathbb{Z}/\ell) = 0$, so by Lemma 2.3, $CH^m(X, 2m; \mathbb{Z}/\ell) \cong CH^m(U, 2m; \mathbb{Z}/\ell)$. By Lemma 2.4, $CH^m(X, 2m; \mathbb{Z}/\ell) \cong CH^m(k(X), 2m; \mathbb{Z}/\ell)$. Now, using the commutative diagram in the proof of Theorem 3.1 (with $n = 2m$), an identical argument enables us to conclude that $\sigma^* : CH^m(k, 2m; \mathbb{Z}/\ell) \to CH^m(X, 2m; \mathbb{Z}/\ell)$ is an isomorphism. By Theorem 2.5 or Theorem 2.6, $CH^m(k, 2m; \mathbb{Z}/\ell) = H^0_{\text{ét}}(k, \mathbb{Z}/\ell(m)) \cong \mathbb{Z}/\ell$.  

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Thus, the middle term in the universal coefficients sequence is isomorphic to $\mathbb{Z}/\ell$; to show $CH^m(X, 2m) = 0$, it suffices to show that $\iota CH^m(X, 2m - 1) = \mathbb{Z}/\ell$. Indeed, by functoriality it suffices to show $\iota CH^m(k, 2m - 1) = \mathbb{Z}/\ell$.

Again, by Theorem 2.5 or Theorem 2.6, $CH^m(k, n; \mathbb{Z}/\ell) = H_{\text{et}}^{2m-n}(k, \mathbb{Z}/\ell(m))$, which vanishes unless $n = 2m$. Thus, by the universal coefficients sequence, $CH^m(k, n)$ is $\ell$-torsion-free as long as $n \neq 2m - 1$. As argued in [PW], Proposition 1.3 and Lemma 1.4, all differentials in the Bloch-Lichtenbaum spectral sequence

$$E_2^{p,q} = CH^q(k, -p - q) \Rightarrow K_{-p-q}(k)$$

are zero; thus, $CH^m(k, 2m - 1)_{\text{tors}} \cong K_{2m-1}(k)_{\text{tors}}$. In particular, by the Quillen-Lichtenbaum conjecture, we see that $\iota CH^m(k, 2m - 1) = \ell K_{2m-1}(k) = \mathbb{Z}/\ell$.

It is now relatively easy to develop analogous results for varieties over fields which contain an algebraically closed field.

**Corollary 3.6.** Let $k \subseteq L$ be fields with $k$ algebraically closed. Suppose $\ell$, $m$, and $n$ are as in Theorem 3.1. Then $CH^m(L, n; \mathbb{Z}/\ell) = 0$.

**Proof.**

If $L$ is finitely generated over $k$, then $L = k(X)$ for some variety $X$ over $k$ and the conclusion follows from Theorem 3.1. In general, write $L = \varprojlim L_i$, where $k \subseteq L_i \subseteq L$ and $L_i$ is finitely generated over $k$. Because higher Chow groups commute with direct limits, we have:

$$CH^m(L, n; \mathbb{Z}/\ell) = \varprojlim \ CH^m(L_i, n; \mathbb{Z}/\ell) = 0.$$

**Corollary 3.7.** Let $k \subseteq L$ be fields with $k$ algebraically closed. Then for $m \geq 1$, $CH^m(L, 2m)$ is uniquely divisible.

**Proof.**

Write $L$ as a direct limit of fields finitely generated over $k$. Since each such field $F$ is finitely generated over $k$, it is the field of rational functions of some variety $Y$ over $k$. Now, $CH^m(F, 2m) = CH^m(k(Y), 2m)$ is the direct limit of $CH^m(U, 2m)$ over all neighborhoods $U$ of the generic point of $Y$. By Corollary 3.5, each $CH^m(U, 2m)$ is uniquely divisible, so the same must be true of $CH^m(F, 2m)$ and hence of $CH^m(L, 2m)$.

**Corollary 3.8.** Let $L$ be any field containing the algebraic closure $k$ of its prime field. Suppose $X$ is a quasiprojective scheme over $L$, with $\ell$, $m$, and $n$ as in Theorem 3.1. Then $CH^m(X, n; \mathbb{Z}/\ell) = 0$. 7
Proof.
By Lemma 3.2 and the preceding remark, we may assume that $X$ is integral. The proof now proceeds by induction on $d = \dim X$. The case $d = 0$ is covered by Corollary 3.6. Now suppose $d > 0$. Let $j : U \hookrightarrow X$ be an open subset with $Z = X - U$ of codimension 1. By induction, $CH^{m-1}(Z, n; \mathbb{Z}/\ell) = CH^{m-1}(Z, n-1; \mathbb{Z}/\ell) = 0$, so by Lemma 2.3 $CH^m(U, n; \mathbb{Z}/\ell)$, and so by Lemma 2.4, $CH^m(U, n; \mathbb{Z}/\ell) \cong CH^m(L(X), n; \mathbb{Z}/\ell)$. By Corollary 3.6, $CH^m(L(X), n; \mathbb{Z}/\ell) = 0$, which completes the induction.

The analogue of Corollary 3.5 also holds when the base field contains an algebraically closed field:

**Corollary 3.9.** With notation as in Corollary 3.8, the group $CH^m(X, n)$ is torsion-free when $n = 2m$ and uniquely divisible when $n > 2m$. When $m \geq 1$, $CH^m(X, 2m)$ is uniquely divisible.

**Proof.**
Again, the only statement which requires proof is divisibility of $CH^m(X, 2m)$. Let $U \subseteq X$ be an open set with $Z = X - U$ of codimension 1. By the assertion already proven, the terms $CH^{m-1}(Z, 2m)$ and $CH^{m-1}(Z, 2m-1)$ are uniquely divisible, so by Lemma 2.3 it suffices to prove that $CH^m(U, 2m)$ is divisible. By Lemma 2.4 and the usual direct limit argument, it suffices to show that $CH^m(L(X), 2m)$ is divisible. This follows from Corollary 3.7.

### 4 Connection to the Beilinson-Soulé conjecture

A well-known conjecture due to Beilinson and Soulé might be phrased as stating that for any smooth variety $X$ over a field $k$, $CH^m(X, n) = 0$ when $n > 2m$. Corollary 3.8 might be interpreted as a “mod $\ell$” version of the Beilinson-Soulé conjecture. In particular, it implies:

**Proposition 4.1.** Let $L$ be a field containing the algebraic closure of its prime field and $X$ a quasiprojective scheme over $L$. Suppose $n > 2m$. Then

$$CH^m(X, n) = 0 \text{ if and only if } CH^m(X, n; \mathbb{Q}) = 0$$

This is of interest because the Riemann-Roch theorem in higher $K$-theory identifies $CH^m(X, n; \mathbb{Q})$ with the graded piece $K^{(m)}_n(X) \otimes \mathbb{Q}$ arising from the $\gamma$-filtration on $K_n(X) \otimes \mathbb{Q}$. In particular, if the groups $K_n(X)$ are known to be torsion for $n \geq 1,$
one may deduce the vanishing of the (integral) higher Chow groups. This is the case, for example, if $p$ is a prime and $X$ is a variety of dimension $\leq 3$ in the class $A(\overline{\mathbb{F}}_p)$ defined by Soulé [So]; this class is closed under products, and includes all curves, abelian varieties, and unirational varieties of degree at least 3. Parshin conjectured that $K_i(X)$ is torsion for $i \geq 1$ and any smooth projective $X$ over $\overline{\mathbb{F}}_p$; hence that all such $X$ belong to $A(\overline{\mathbb{F}}_p)$. Thus we have the following:

**Proposition 4.2.** If $X$ satisfies Parshin’s conjecture, then $CH^m(X, n) = 0$ for all $m \geq 1$ and $n \geq 2m$.

**Proof.**
The hypothesis implies that $K_n(X)$ is torsion for all $n \geq 1$, so by the Riemann-Roch Theorem, the groups $CH^m(X, n)$ are torsion when $n \geq 1$. By Corollary 3.5, these groups are torsion-free when $m \geq 1$, $n \geq 2m$, so we conclude that $CH^m(X, n) = 0$. 

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References


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