Zero-cycles on supersingular abelian varieties, revisited

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Abstract

Let $k$ be a field of positive characteristic and $A$ a supersingular abelian variety over $k$; it is known that there is a supersingular elliptic curve $E$ such that $A$ is isogenous (over $\bar{k}$) to $E^g$, where $g = \dim A$. We give a description of the higher Chow groups of zero-cycles on $A$ in terms of those on $E$, generalizing a result of Maruyama and Suwa on the Albanese kernel.

1 Introduction

Let $k$ be a field of positive characteristic. An elliptic curve $E$ over $k$ is called supersingular if its endomorphism ring $\text{End}_{\bar{k}}(E)$ (over $\bar{k}$) is isomorphic to an order in a definite quaternion $\mathbb{Q}$-algebra. (In general, $\text{End}_{\bar{k}}(E)$ is a free abelian group of rank 1, 2, or 4) If $A$ is an abelian variety over $k$, the corresponding notion of supersingularity for $A$ is equivalent to requiring that $A$ be isogenous over $\bar{k}$ to $E^g$, where $E$ is an elliptic curve over $\bar{k}$ and $g = \dim A$.

Mixed $K$-groups, a generalization of Milnor $K$-groups, were introduced by the author in [A1] and studied further in [A2] and [A3]. The key technical result of this paper is that if $E$ is a supersingular elliptic curve over a field $k$, then the group $K(k; \ldots, E, E, \ldots)$ is of finite exponent. (The ellipses represent any other arguments of the mixed $K$-group which may be present.) Putting this result together with some of our previous work relating mixed $K$-groups to higher Chow groups of zero-cycles [A1], [A2] we prove various results on the structure of the higher Chow groups of cycles on $A$.

The motivation for considering such questions goes back to a result of Maruyama and Suwa [MS] that the Albanese kernel of a supersingular abelian variety over an algebraically closed field is trivial. Working rationally, Fakhruddin [Fa] has shown more recently that all the Chow groups of a supersingular variety have a relatively
uncomplicated structure. Both these approaches, however, rely on the fact that the ground field is algebraically closed, in order to make use of tools such as the Abel-Jacobi map. Of course, if one works over an arbitrary field $k$, then by writing $\overline{k}$ as a direct limit of finite extensions of $k$, a simple norm argument shows that the Albanese kernel of a supersingular abelian variety over $k$ must be torsion.

This investigation takes a rather different approach to such questions, using nothing more technical than mixed $K$-groups and the structure of the endomorphism ring of a supersingular elliptic curve. Working over an arbitrary field, we prove the stronger result that the Albanese kernel of a supersingular abelian variety is a group of finite exponent, and recover the Maruyama-Suwa vanishing result in the case that the ground field is algebraically closed. In addition to obtaining results on the Albanese kernel, we also obtain results about analogous subgroups of the higher Chow groups.

The primary results may be collected together as follows:

**Theorem.**

Let $k$ be a field of positive characteristic and $A$ be a supersingular abelian variety of dimension $g$ over $k$. Let $s \geq 0$ be any integer, and $E$ a supersingular elliptic curve over $k$ such that $A$ is isogenous (over $k$) to $E^g$. Then there is an exact sequence

$$0 \to F_1 \to CH^{g+s}(A, s) \to K^M_s(k) \oplus K_s(k; E, G_m) \oplus \Gamma \oplus F \to F_2 \to 0$$

where $K^M_s(k)$ is the $s$th Milnor $K$-group, $K_s(k; E, G_m)$ is a mixed $K$-group, and $F, F_1, F_2$ are groups of finite exponent. When $k$ is algebraically closed, $F = F_1 = F_2 = 0$.

(One may obtain a more general but less elegantly stated result if one drops the assumption that $A$ is isogenous to $E^g$ over $k$, assuming rather that the isogeny, and possibly $E$ itself, is defined over some extension of $k$.) The proof of this theorem is encumbered by numerous technical lemmas, but the strategy is readily described: for convenience, we say that two abelian groups $G_1, G_2$ are isogenous if there is a homomorphism between them with finite kernel and cokernel. Given a supersingular abelian variety $A$ over $k$, show that we may replace $k$ by a finite extension over which $A$ is isogenous to $E^g$ for some supersingular elliptic curve $E$ over $k$ (Lemma 6.1). Then, show that $CH^{g+s}(A, s)$ is isogenous to $CH^{g+s}(E^g, s)$ (Proposition 6.2). Next, identify $CH^{g+s}(E^g, s)$ with the mixed $K$-group $K_s(k; CH_0(E), \ldots, CH_0(E); G_m)$ (Theorem 4.2), and use Theorem 3.4 to write this mixed $K$-group as $K^M_s(k) \oplus K_s(k; E, G_m) \oplus H$, where $H$ is a direct sum of mixed $K$-groups of the form $K(k; \ldots, E, E, \ldots)$. The
most interesting part of the proof is the last step (Theorem 8.1), which is to show that groups of the form $K(k; \ldots, E, E, \ldots)$ have finite exponent.

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2 Mixed $K$-groups

Following [A1], we define mixed $K$-groups. Let $k$ be a field. If $X$ is a smooth quasiprojective variety defined over $k$, we denote by $CH_0(X)$ for the group of zero-cycles on $X$ modulo rational equivalence. If $G$ is a group scheme defined over $k$ and $A$ is a $k$-algebra, we use the notation $G(A)$ for the group of $A$-rational points, i.e. morphisms $\text{Spec} A \to G$ which commute with the structure maps.

Let $r \geq 0$ and $s \geq 0$ be integers; let $X_1, \ldots, X_r$ be smooth quasiprojective varieties defined over $k$ and $G_1, \ldots, G_s$ semi-abelian varieties defined over $k$. Set

$$T = \bigoplus_{E/k \text{ finite}} CH_0((X_1)_E) \otimes \cdots \otimes CH_0((X_r)_E) \otimes G_1(E) \otimes \cdots \otimes G_s(E)$$

We use the notation $(a_1 \otimes \cdots \otimes a_r \otimes b_1 \otimes \cdots \otimes b_s)_E$ to refer to a homogeneous element living in the direct summand of $F$ corresponding to the field $E$.

Define $R \subseteq T$ as the subgroup generated by the elements of the following type:

- **M1.** For convenience of notation, set $H_i(E) = CH_0((X_i)_E)$ for $i = 1, \ldots, r$ and $H_j(E) = G_{j-r}(E)$ for $j = r + 1, \ldots, r + s$.

For every diagram $k \hookrightarrow E_1 \xhookrightarrow{\phi} E_2$ of finite extensions of $k$, all choices $i_0 \in \{1, \ldots, r + s\}$ and all choices $h_{i_0} \in H_{i_0}(E_2)$ and $h_i \in H_i(E_1)$ for $i \neq i_0$, the element $R_1(E_1; E_2; i_0; h_1, \ldots, h_{r+s})$ defined to be:

$$(\phi^*(h_1) \otimes \cdots \otimes h_{i_0} \otimes \cdots \otimes \phi^*(h_{r+s}))_{E_2} - (h_1 \otimes \cdots \otimes \phi_*(h_{i_0}) \otimes \cdots \otimes h_{r+s})_{E_1}$$

Here we have used the notation $\phi^*$ ($\phi_*$) to denote the pullback (pushforward) map for the Chow group structure on $H_i$ (if $1 \leq i \leq r$) or the group scheme structure on $H_i$ (if $s \leq i \leq r + s$).
Remark.

K

practice of omitting superfluous semicolons; for example, if

G

We will also consider the case in which

K

for

G

We will often be interested in the case

Z

We complete the picture by defining our group to be

M1

elements of the form

h

Finally, when

r+s

0, this is defined to be:

\[
\sum_v (s_v(f_1) \otimes \ldots \otimes s_v(f_r) \otimes g_1(v) \otimes \ldots \otimes \tilde{T}_v(g_{j_0(v)}, h) \otimes \ldots \otimes g_s(v))_{k(v)}
\]

Here

O_v

is the valuation ring of

v
,

s_v : CH_0((X_i)_K) \rightarrow CH_0((X_i)_{k(v)})

is the specialization map for Chow groups (cf. [F], 20.3), and

g_i(v) \in G_i(k(v))

denotes the reduction of

g_i \in G_i(O_v).

The notation

\tilde{T}_v

tests to the “extended tame symbol” as defined in [So]. In this paper, we will be concerned only with the case that the

G_i

are either abelian varieties or the multiplicative group scheme

G_m.

If

A = G_{j_0(v)}

is an abelian variety, then

\tilde{T}_v(a, h) = ord_v(h^\gamma a(v))

where

ord_v

tests to the ordinary tame symbol; that is, the residue of the element

(-1)^{ord_v(\gamma)ord_v(h)}g^{ord_v(\gamma)h^{-ord_v(\gamma)}} \in O_v^*

in the residue field

k(v)^*.

If

s = 0,

the element

R_2(K; h; f_1, \ldots, f_r; g_1, \ldots, g_s)

is defined to be:

\[
\sum_v \ord_v(h)(s_v(f_1) \otimes \ldots \otimes s_v(f_r))_{k(v)}
\]

Finally, when

r+s > 0,

we define the mixed

K-group,

K(k; CH_0(X_1), \ldots, CH_0(X_r); G_1, \ldots, G_s)

as the quotient

T/R.

We denote the class of a generator

(h_1 \otimes \ldots \otimes h_{r+s})_E

by

\{h_1, \ldots, h_{r+s}\}_E/k.

We refer to the classes of elements of the form

M1

and

M2

as relations of the mixed

K-group.

We complete the picture by defining our group to be

Z

in the case

r = s = 0.

We will often be interested in the case

G_1 = \ldots = G_s = G_m;

hence we use

K_s(k; CH_0(X_0), \ldots, CH_0(X_r); G_m)

as shorthand for

K(k; CH_0(X_0), \ldots, CH_0(X_r); \bigotimes_{i=1}^{s} G_m).

We will also consider the case in which

G_1 = \ldots = G_t = E

for some elliptic curve

E

over

k

and

G_{t+1}, \ldots, G_{t+s} = G_m.

In this context, we write

K_{t,s}(k; CH_0(X_0), \ldots, CH_0(X_r); E; G_m)

for

K(k; CH_0(X_0), \ldots, CH_0(X_r), E, \ldots, E, \bigotimes_{i=1}^{t} G_m, \ldots, G_m).

We also adopt the practice of omitting superfluous semicolons; for example, if

r = 0,

we simply write

K(k; G_1, \ldots, G_s)

for the above group.

Remark.
If $\sigma : Y \rightarrow \text{Spec } k$ is a projective variety, we can define the degree map $\deg = \sigma_* : \text{CH}_0(Y) \rightarrow \text{CH}_0(\text{Spec } k) \cong \mathbb{Z}$ by push-forward of cycles. We define $A_0(Y) := \ker \deg$, and note that if $Y$ contains a $k$-rational point, or more generally if $Y$ admits a zero-cycle of degree 1, then the degree map splits and we have a direct sum decomposition $\text{CH}_0(Y) \cong \mathbb{Z} \oplus A_0(Y)$.

Returning to our situation, suppose that $X_1, \ldots, X_q$ and $Y_1, \ldots, Y_r$ are smooth and quasiprojective, with $Y_1, \ldots, Y_r$ actually projective. By replacing $\text{CH}_0$ with $A_0$ in the appropriate instances, we may define groups $K(k; \text{CH}_0(X_1), \ldots, \text{CH}_0(X_q); A_0(Y_1), \ldots, A_0(Y_r); G_1, \ldots, G_s)$ as was done previously.

It follows easily from the definitions that if $\sigma$ is any permutation of $\{1, \ldots, q\}$, $\tau$ is any permutation of $\{1, \ldots, r\}$, and $\nu$ is any permutation of $\{1, \ldots, s\}$, then there is an obvious isomorphism $K(k; \text{CH}_0(X_1), \ldots, \text{CH}_0(X_q); A_0(Y_1), \ldots, A_0(Y_r); G_1, \ldots, G_s) \cong K(k; \text{CH}_0(X_{\sigma(1)}), \ldots, \text{CH}_0(X_{\sigma(q)}); A_0(Y_{\tau(1)}), \ldots, A_0(Y_{\tau(r)}); G_{\nu(1)}, \ldots, G_{\nu(s)})$.

### 3 Functoriality

We summarize here some functoriality properties of mixed $K$-groups. A more comprehensive treatment of these, together with proofs, may be found in [A2] or in the author’s Ph.D. dissertation.

The first functoriality property deals with change of base field.

**Theorem 3.1.** The groups $K(k; \text{CH}_0(X_1), \ldots, \text{CH}_0(X_q); A_0(Y_1), \ldots, A_0(Y_r); G_1, \ldots, G_s)$ are covariant functorial for arbitrary extensions $k \hookrightarrow k'$ of the base field and contravariant functorial for finite extensions $k \hookrightarrow L$.

The next result covers functoriality in the various varieties involved.

**Theorem 3.2.** The groups $K(k; \text{CH}_0(X_1), \ldots, \text{CH}_0(X_q); A_0(Y_1), \ldots, A_0(Y_r); G_1, \ldots, G_s)$ are covariant functorial for proper maps $X_i \rightarrow X'_i$ and $Y_j \rightarrow Y'_j$, and contravariant functorial for flat maps $X_i \rightarrow X'_i$ and $Y_j \rightarrow Y'_j$; these maps are induced by the corresponding maps on Chow groups. Moreover, if $f_i : G_i \rightarrow G'_i, 1 \leq i \leq s$ are morphisms of semi-abelian varieties over $k$, then there is an induced map:

$$(f_1, \ldots, f_s)_* : K(k; \text{CH}_0(X_1), \ldots, \text{CH}_0(X_q); A_0(Y_1), \ldots, A_0(Y_r); G_1, \ldots, G_s) \rightarrow K(k; \text{CH}_0(X_1), \ldots, \text{CH}_0(X_q); A_0(Y_1), \ldots, A_0(Y_r); G'_1, \ldots, G'_s)$$

We also record the following useful fact, which appears as Lemma 2.6 in [A2].
Lemma 3.3. With notation as above, there is a natural map inducing an isomorphism:

\[ K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_q); A_0(Y_1), \ldots, A_0(Y_r); G_1, \ldots, G_s) \cong K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_q); A_0(Y_1), \ldots, A_0(Y_r); G_1, \ldots, G_s) \]

The relationship between various groups of this type is illustrated by the following useful decomposition theorem:

Theorem 3.4. ([A2], Cor. 2.9) With \( k, X_i, Y_i \) and \( G_i \) as before, let \( Z_1, \ldots, Z_t \) be smooth projective varieties defined over \( k \), each admitting a zero-cycle of degree 1. To ease notation, write \( K_{i_1, \ldots, i_t} \) for the group \( K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_1); A_0(Z_{i_1}), \ldots, A_0(Z_{i_t}); A_0(Y_1), \ldots, A_0(Y_r); G_1, \ldots, G_s) \). Then there is a natural map inducing an isomorphism:

\[ K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_q), \mathcal{CH}_0(Z_1), \ldots, \mathcal{CH}_0(Z_t); A_0(Y_1), \ldots, A_0(Y_r); G_1, \ldots, G_s) \cong \bigoplus_{0 \leq i \leq t} \bigoplus_{i_1 < \ldots < i_t \leq t} K_{i_1, \ldots, i_t} \]

4 Relationship to Higher Chow Groups

There is a close relationship between certain of the higher Chow groups (as defined by Bloch [Bl3]) and the mixed \( K \)-groups discussed above. One of the main results of [A1] is the following:

Theorem 4.1. ([A1], Theorem 6.1) Let \( k \) be a field, \( s \geq 0 \) an integer, and \( X \) a smooth projective variety over \( k \) of dimension \( d \). Then there is a natural map inducing an isomorphism:

\[ \alpha = \alpha_X : CH^{d+s}(X, s) \cong K_s(k; \mathcal{CH}_0(X); \mathbb{G}_m) \]

We wish to generalize this as follows:

Theorem 4.2. Let \( k \) be a field, \( s \geq 0 \) an integer, and \( X_1, X_2 \) smooth projective varieties over \( k \) of dimensions (respectively) \( d_1, d_2 \). Let \( d = d_1 + d_2 = \dim(X_1 \times_k X_2) \). Then there is a natural map inducing an isomorphism:

\[ \alpha = \alpha_X : CH^{d+s}(X, s) \cong K_s(k; \mathcal{CH}_0(X_1), \mathcal{CH}_0(X_2); \mathbb{G}_m) \]
It is easy to see how Theorem 4.1 may be deduced from Theorem 4.2 and Lemma 3.3. It seems that the obvious way to prove the converse would be to construct mutually inverse maps between the two mixed $K$-groups described above. Unfortunately, this proves to be problematic, at least when $k$ is not algebraically closed: it does not seem at all clear \textit{a priori} that the obvious candidate for a map $K(k;\mathcal{CH}_0(X);\mathbb{G}_m) \to K(k;\mathcal{CH}_0(X_1),\mathcal{CH}_0(X_2),\mathbb{G}_m)$ – that is, the one induced by projection on the factors – is well-defined; one encounters trouble trying to prove that relations of type M1 are killed by this rule.

Instead, we prove Theorem 4.2 directly here, but refer the reader to [A1] for some of the technical details in the proof. The casual reader would be well advised to skip the remainder of this section.

**Proof of Theorem 4.2**

If $E/k$ is a finite extension, we let $p_{1,E}^*, p_{2,E}^*$ denote the maps associated to the projections on the first and second factors of $(X_1)_E \times_E (X_2)_E$.

We begin by constructing a map

$$\beta : K_s(k;\mathcal{CH}_0(X_1),\mathcal{CH}_0(X_2);\mathbb{G}_m) \to CH^{d+s}(X, s).$$

Let $\{z_1, z_2, a_1, \ldots, a_s\}_{E/k}$ be a generator of the group on the left, with $\phi_{E/k} : \text{Spec } E \to \text{Spec } k$ the map induced by the inclusion $k \hookrightarrow E$. Under the identification $E^* \cong CH^1(E, 1)$, we may then define

$$\beta(\{z_1, z_2, a_1, \ldots, a_s\}_{E/k}) = (\phi_{E/k})_*(p_{1,E}^*)y_1 \cdot (p_{2,E}^*)z_2 \cdot a_1 \cdot \ldots \cdot a_s$$

Thus, if $k \hookrightarrow E \hookrightarrow F$ are finite extensions and $y_1 \in CH_0(X_F), z_2 \in CH_0(X_E)$ and $a_1, \ldots, a_s \in E^*$, we have

$$\beta(\{y_1, \phi_{F/E}^*z_2; \phi_{F/E}^*a_1, \ldots, \phi_{F/E}^*a_s\}_{F/k})$$

$$= (\phi_{F/k})_*(\phi_{F/E})_*((p_{1,F}^*)y_1 \cdot (p_{2,F}^*)z_2 \cdot \phi_{F/E}^*a_1 \cdot \ldots \cdot \phi_{F/E}^*a_s)$$

$$= (\phi_{E/k})_*(\phi_{F/E})_*((p_{1,E}^*)y_1 \cdot \phi_{F/E}^*((p_{2,E}^*)z_2 \cdot \phi_{F/E}^*a_1 \cdot \ldots \cdot \phi_{F/E}^*a_s))$$

$$= (\phi_{E/k})_*(\phi_{F/E})_*((p_{1,E}^*)y_1 \cdot (p_{2,E}^*)z_2 \cdot a_1 \cdot \ldots \cdot a_s)$$

$$= (\phi_{F/E})_*((p_{1,E}^*)y_1 \cdot (p_{2,E}^*)z_2 \cdot a_1 \cdot \ldots \cdot a_s)$$
This argument, modified appropriately to cover the other cases, shows that relations of type \( M_1 \) are killed by \( \beta \). Now suppose \( R = R_2(K; h; y_1, y_2; g_1, \ldots, g_s) \) is a relation of type \( M_2 \). Then the proof of Theorem 4.1 shows that \( s_v((p_1^K)^*y_1 \cdot (p_2^K)^*y_2) \cdot g_1 \cdot g_s \cdot h = \partial_v((p_1^K)^*y_1 \cdot (p_2^K)^*y_2 \cdot g_1 \cdot g_s \cdot h) \), where \( \partial_v : CH^{d+s+1}(X_K, s + 1) \rightarrow CH^{d+s}(X_{k(v)}, s) \) is the boundary map arising from the localization sequence. However, since \( s_v \) preserves intersection product ([F], Corollary 20.3) and commutes with pullbacks ([F], Proposition 20.3), we have:

\[
\beta(R) = \sum_v (\phi_{k(v)/k})_* ((p_1^{k(v)})^* s_v(y_1) \cdot (p_2^{k(v)})^* s_v(y_2) \cdot g_1 \cdot g_s \cdot h)
\]

\[
= \sum_v (\phi_{k(v)/k})_* (s_v((p_1^K)^*y_1) \cdot s_v((p_2^K)^*y_2) \cdot g_1 \cdot g_s)
\]

\[
= \sum_v (\phi_{k(v)/k})_* (s_v((p_1^K)^*y_1 \cdot (p_2^K)^*y_2) \cdot g_1 \cdot g_s)
\]

\[
= \sum_v (\phi_{k(v)/k})_* \partial_v((p_1^K)^*y_1 \cdot (p_2^K)^*y_2 \cdot g_1 \cdot g_s \cdot h)
\]

\[
= 0
\]

by the Reciprocity Law for higher Chow groups. (cf. [A1], Theorem 4.5).

We continue by constructing a map \( \alpha : CH^{d+s}(X, s) \rightarrow K_s(k; CH_0(X_1), CH_0(X_2); G_m) \).

Since the group \( CH^{d+s}(X, s) \) consists of zero-cycles, it is generated by classes \([P]\), where \( P : \text{Spec } k(P) \rightarrow X \times_k \partial k \) is a closed point. By definition of fibered product, we see that \( P \) is determined by the data \( x_1 : \text{Spec } k(P) \rightarrow X_1, x_2 : \text{Spec } k(P) \rightarrow X_2 \) and \( a_1, \ldots, a_s \in k(P)^* \). Now define

\[
\alpha : CH^{d+s}(X, s) \xrightarrow{\alpha} K_s(k; CH_0(X_1), CH_0(X_2); G_m)
\]

by

\[
[P] \mapsto \{([x_1]_{k(P)}), ([x_2]_{k(P)}), a_1, \ldots, a_s \}_{k(P)/k}
\]

The following lemma is a generalization of [A1], Lemma 3.5.

**Lemma 4.3.** Suppose \( Q_1 \in CH_0(X_1), Q_2 \in CH_0(X_2) \).
For every finite extension $F/k$, there exists a well-defined homomorphism $j_F(Q_1, Q_2) : K_s(F; \mathbb{G}_m) \to K_s(F; CH_0((X_1)_F), CH_0((X_2)_F); \mathbb{G}_m)$ taking a generator $\{b_1, \ldots, b_s\}_{M/F}$ to the element $\{[Q_1]_M, [Q_2]_M, b_1, \ldots, b_s\}_{M/F}$.

Let $k \hookrightarrow F_1 \hookrightarrow F_2$ be a diagram of finite extensions of $k$.

Then the following diagram commutes:

$$
\begin{array}{ccc}
K_s^M(F_2) & \xrightarrow{\gamma_{F_2}} & K_s(F_2; \mathbb{G}_m) \\
\downarrow N_{F_2/F_1}^M & & \downarrow N_{F_2/F_1}^M \\
K_s^M(F_1) & \xrightarrow{\gamma_{F_1}} & K_s(F_1; \mathbb{G}_m)
\end{array}
$$

In terms of our previous notation, then we have

$$
\alpha([P]) = N_{k(P)/k}^M(j_{k(P)}([x_1]_{k(P)}), [x_2]_{k(P)})(\gamma_{k(P)}\{a_1, \ldots, a_s\})
$$

To check that this rule is well-defined, we must show that elements of $d_{s+1}(z^{d+s}(X, s + 1))$ are killed by $\alpha$.

The generators of the group $z^{d+s}(X, s + 1)$ correspond to well-positioned dimension 1 subvarieties $C \subseteq X \times_k \Box_k^{s+1}$. This means that for any face $F$ of $\Box_k^{s+1}, C$ intersects $X \times_k F$ in dimension 0 (i.e. in points) if $F$ has codimension 1 and does not meet $X \times_k F$ at all if $F$ has codimension $\geq 2$. Let $\nu : \tilde{C} \to C$ be the normalization of $C$.

The inclusion $i : C \subseteq X \times_k \Box_k^{s+1}$ is defined by a collection of $s + 2$ morphisms: $f : C \to X, g_i : C \to \Box_k^{i}, i = 1 \ldots s + 1$. Likewise, the composition $\tilde{i} = \nu \circ i : \tilde{C} \to X$ is defined by $s + 2$ morphisms: $\tilde{f} : \tilde{C} \to X, \tilde{g}_i : \tilde{C} \to \Box_k^{i} \to \Box_k$.

The assumption that $C$ meets the codimension 1 faces of the cube in points translates into the fact that none of the $g_i$ are identically 0 or $\infty$. The condition that $C$ does not meet the codimension $\geq 2$ faces of the cube at all implies that given any $x \in C^1$, at most one of the $g_i$ assumes a value of $0$ or $\infty$ at $x$. In particular, for every $v \in \tilde{C}^1$, at most one of the $\tilde{g}_i$ assumes a value of 0 or $\infty$ at $v$.

For $x \in C^1$, define $j(x)$ to be the index $j$ such that $g_j(x) \in \{0, \infty\}$ if such an index exists, or 1 if not such index exists. Likewise, for $v \in \tilde{C}^1$, define $\tilde{j}(v)$ to be the index $\tilde{j}$ such that $\tilde{g}_\tilde{j}(v) \in \{0, \infty\}$ if such an index exists, or 1 if no such index exists.

By definition of the boundary map $d_{s+1}$, we have:

$$
d_{s+1}(C) = \sum_{x \in C^1} (-1)^{j(x)\text{ord}_x(g_j(x))}(f(x) \times g_1(x) \times \ldots \times g_{j(x)}(x) \times \ldots \times g_{s+1}(x))
$$
Now let $D$ denote the proper smooth model for $k(C)$ over $k$. Since $\hat{C}$ is normal, it may be identified with a subset of $D$. The functions $\tilde{g}_i : \hat{C} \to \mathbb{P}^1_k$ may thus be considered rational functions on $D$, which of course extend to morphisms which we denote (somewhat abusively) by $\tilde{g}_i : D \to \mathbb{P}^1_k$. Furthermore, since $X$ is projective, $\tilde{f} : \hat{C} \to X$ extends to a morphism $\tilde{f} : D \to X$.

**Lemma 4.4.** Suppose $v \in D - \hat{C}$. Then there exists $i(v) \in \{1, \ldots, s + 1\}$ such that $\tilde{g}_{i(v)}(v) = 1$.

**Proof.**

Let $K$ denote the function field $k(C) = k(\hat{C}) = k(D)$. If $\tilde{g}_i(v) \neq 1$ for all $i = 1, \ldots, s + 1$, then all the $\tilde{g}_i$ are regular at $v \in D$, and therefore (by completeness of $X$) we have a morphism $\text{Spec } \mathcal{O}_v \xrightarrow{\phi_v} X \times_k \square_{k}^{s+1}$. Putting all of this data into one diagram, we have:

$$
\begin{array}{c}
\text{Spec } K \\
\downarrow
\end{array} \xrightarrow{\phi_v} \hat{C} \xrightarrow{\tilde{f} \times \tilde{g}_1 \times \ldots \times \tilde{g}_{s+1}} \text{Spec } \mathcal{O}_v \xrightarrow{\phi_v} X \times_k \square_{k}^{s+1}
$$

Now the right vertical map is proper, so the valuative criterion for properness gives a (unique) map $\text{Spec } \mathcal{O}_v \to \hat{C}$ making the resulting diagram commutative. This is a contradiction, because $v$ was chosen to be in $D - \hat{C}$.

Let $p_1 : X_1 \times_k X_2 \to X_1$ and $p_2 : X_1 \times_k X_2 \to X_2$ denote the projection maps, and set $f_1 = p_1 \circ f$, $f_2 = p_2 \circ f$, $\hat{f}_1 = p_1 \circ \hat{f}$, $\hat{f}_2 = p_2 \circ \hat{f}$.

Resuming our calculation, we note:

$$
\alpha(d_{s+1}(C)) = \alpha\left(\sum_{x \in C^1} (-1)^{j(x)} \text{ord}_x(g_{j(x)}) \{f(x) \times g_1(x) \times \ldots \times g_{j(x)}(x) \times \ldots \times g_{s+1}(x)\}\right)
$$

$$
= \sum_{x \in C^1} (-1)^{j(x)} \text{ord}_x(g_{j(x)}) \{[f_1(x)_{k(x)}], [f_2(x)_{k(x)}], g_1(x), \ldots, g_{j(x)}(x), \ldots, g_{s+1}(x)\}_{k(x)/k}
$$

By [F], ex. 1.2.3, we have $\text{ord}_v(g_{j(x)}) = \sum_{\nu(v) = v} [k(v) : k(x)] \text{ord}_v(\tilde{g}_{j(v)})$. (Of course, $g_{j(x)} = \tilde{g}_{j(v)}$ as elements of $k(C) = k(\hat{C})$) Thus, the above expression equals:

$$
= \sum_{v \in C^1} (-1)^{\tilde{j}(v)} [k(v) : k(\nu(v))] \text{ord}_v(\tilde{g}_{j(v)}) \{[\tilde{f}_1(v)_{k(v)}], [\tilde{f}_2(v)_{k(v)}], \tilde{g}_1(v), \ldots, \tilde{g}_{j(v)}(v), \ldots, \tilde{g}_{s+1}(v)\}_{k(\nu(v))/k}
$$
Using a relation of type $M1$, this may be identified with:

$$\sum_{v \in \mathcal{C}_1} (-1)^{\beta(v)} \text{ord}_v(\tilde{g}_j(v))\{[\tilde{f}_1(v)_{k(v)}], [\tilde{f}_2(v)_{k(v)}], \tilde{g}_1(v), \ldots, \tilde{g}_{j(v)}(v), \ldots, \tilde{g}_{s+1}(v)\}_{k(v)/k}$$

By Lemma 4.4, this is the same as the above sum taken over all valuations of $k(C)$ fixing $k$:

$$\sum_{v \in D} (-1)^{\beta(v)} \text{ord}_v(\tilde{g}_j(v))\{[\tilde{f}_1(v)_{k(v)}], [\tilde{f}_2(v)_{k(v)}], \tilde{g}_1(v), \ldots, \tilde{g}_{j(v)}(v), \ldots, \tilde{g}_{s+1}(v)\}_{k(v)/k}$$

Letting $\phi : \text{Spec } k(C) \rightarrow X$ denote the map naturally induced by $f$, we see at last that this is a relation $R_2(k(C); g_{s+1}; p_1 \circ \phi_{k(C)}; p_2 \circ \phi_{k(C)}; g_1, \ldots, g_s)$ of the group $K_s(k; CH_0(X_1), CH_0(X_2); G_m)$. This concludes the proof that $\alpha$ is well-defined.

It remains to check that the compositions $\beta \circ \alpha$ and $\alpha \circ \beta$ are the identity maps on the appropriate groups. We begin with the former.

It suffices to verify the assertion on generators $[P]$ of $CH^{d+s}(X, s)$ corresponding to closed points $P : \text{Spec } k(P) \rightarrow X \times_k \mathbb{A}_{k}^s$. As in the proof of the well-definedness of $\alpha$, we observe that $P$ is determined by $y_1 : \text{Spec } k(P) \rightarrow X_1$, $y_2 : \text{Spec } k(P) \rightarrow X_2$, and elements $a_1, \ldots, a_s \in k(P)^*$.

By the product structure on the cubical complex, we have:

$$[P_{k(P)}] = [y_{k(P)}] \times [a_1] \times \ldots \times [a_s]$$

viewing $[y_{k(P)}]$ as an element of $CH^{d}(X_{k(P)}, 0)$ and $[a_1], \ldots, [a_s]$ as elements of $CH^{1}(k(P), 1)$.

Then, computing directly from the definitions,

$$\beta(\alpha([P])) = \beta([y_1_{k(P)}], [y_2_{k(P)}], a_1, \ldots, a_s)_{k(P)/k}$$

$$= (\phi_{k(P)/k})_*( [y_{k(P)}] \times [a_1] \times \ldots \times [a_s])$$

$$= (\phi_{k(P)/k})_*( [P_{k(P)}])$$

$$= [P]$$

To show that $\alpha \circ \beta = id$, it suffices to show that $\alpha$ is surjective. To this end, fix a generator $g = \{[P_1], [P_2], a_1, \ldots, a_s\}_{E/k} \in K_s(k; CH_0(X); G_m)$, where $P_i \in (X_i)_E$
and \(a_1, \ldots, a_s \in E^*\). Let \(F = k(P_1)\), and \(\phi_{F/E} : E \hookrightarrow F\) the inclusion map. Then 

\[(\phi_{F/E})_*([P_1])_{F} = [P_1],\]

so by a relation of type \(\text{M1}\), we have \([[P_1], [P_2], a_1, \ldots, a_s]_{E/k} = \{[P_1], \phi_{F/E}^*([P_2], \phi_{F/E}^*([a_1], \ldots, \phi_{F/E}^*([a_s]))_{F/k}\}.\] This calculation shows that we may assume without loss of generality that \(P_1\) and \(P_2\) are \(E\)-rational points of \(X_1, X_2\) respectively. Let \(P\) be the \(E\)-rational point of \(X\) whose projection onto \(X_i\) is \(P_i\).

The data \(P, a_1, \ldots, a_s\) define a closed point \(x_E\) of \(X_E \times_E \square^k_E\); consider its image \(x\) under the natural map \(X_E \times_E \square^k_E \rightarrow X \times_k \square^k_k\). Evidently, \(x\) is a closed point \(\text{Spec } L \rightarrow X \times_k \square^k_k\); let \(Q : \text{Spec } L \rightarrow X\) be defined by projection of \(x\) onto \(X\). Note in particular that \(\phi_{E/L}^*[Q_L] = [P]\)

We argue now that the element \(y = [P] \times a_1 \times \ldots \times a_s \in CH^{d+s}(X_E, s)\) has the same image under both compositions in the diagram:

\[
\begin{array}{c}
\xymatrix{CH^{d+s}(X_E, s) \ar[r]^-{\alpha_E} & K_*(E; \mathcal{CH}_0((X_1)_E), \mathcal{CH}_0((X_2)_E); \mathbb{G}_m) \\
\phi_{E/L}^* \ar[u] & N_{E/L}^{MM} \ar[l]}
\end{array}
\]

and that \((\phi_{E/L})_*(y) \in CH^{d+s}(X_L, s)\) has the same image under both compositions in the diagram:

\[
\begin{array}{c}
\xymatrix{CH^{d+s}(X_L, s) \ar[r]^-{\alpha_L} & K_*(L; \mathcal{CH}_0((X_1)_L), \mathcal{CH}_0((X_2)_L); \mathbb{G}_m) \\
\phi_{L/k}^* \ar[u] & N_{L/k}^{MM} \ar[l]}
\end{array}
\]

This will suffice to prove our assertion.

For the first diagram,

\[
N_{E/L}^{MM}(\alpha_E([P] \times a_1 \times \ldots \times a_s))
\]

\[
= \{[P_1], [P_2], a_1, \ldots, a_s\}_{E/L}
\]

Also,

\[
\alpha_L((\phi_{E/L})_*([P] \times a_1 \times \ldots \times a_s)) = \alpha_L((\phi_{E/L})_*([\phi_{E/L}^*[Q_L] \times a_1 \times \ldots \times a_s])
\]

By the projection formula for higher Chow groups,
\[ \alpha_L([Q_L] \cdot (\phi_{E/L})_*(a_1 \cdot \ldots \cdot a_s)) \]

\[ = \alpha_L([Q_L] \cdot T_k(N_{E/L}^M{a_1, \ldots, a_s})) \]

From the definition of \( \alpha_L \),

\[ = j_L[Q](\gamma_L(N_{E/L}^M{a_1, \ldots, a_s})) \]

By Lemma 4.3,

\[ = N_{E/L}^{MM}(j_E[Q](\gamma_E{a_1, \ldots, a_s})) \]

\[ = N_{E/L}^{MM}([P_1], [P_2], a_1, \ldots, a_s)_{E/E} \]

\[ = \{ [P_1], [P_2], a_1, \ldots, a_s \}_{E/L} \]

which completes the verification for the first diagram.

For the second diagram,

\[ \alpha((\phi_{L/k})_*(\phi_{E/L})_*(y)) = \alpha((\phi_{L/k})_*(\phi_{E/L})_*([P] \times a_1 \times \ldots \times a_s)) \]

\[ = \alpha((\phi_{L/k})_*(\phi_{E/L})_*([Q_L] \times a_1 \times \ldots \times a_s)) \]

\[ = \alpha((\phi_{L/k})_*([Q_L] \times (\phi_{E/L})_*([a_1 \cdot \ldots \cdot a_s])) \]

\[ = N_{L/k}^{MM}(j_L([p_1]_*[Q_L], (p_2)^*[Q_L]))(\gamma_L(N_{E/L}^M{a_1, \ldots, a_s})) \]

Also,

\[ N_{L/k}^{MM}(\alpha_L((\phi_{E/L})_*(y))) = N_{L/k}^{MM}(\alpha_L((p_1)_*[Q_L] \cdot (p_2)^*[Q_L] \cdot N_{E/L}^{CH}(a_1 \cdot \ldots \cdot a_s))) \]

\[ = N_{L/k}^{MM}(\alpha_L((p_1)_*[Q_L] \cdot (p_2)^*[Q_L] \cdot N_{E/L}^{CH}(a_1 \cdot \ldots \cdot a_s))) \]

\[ = N_{L/k}^{MM}(j_L([p_1]_*[Q_L], (p_2)^*[Q_L])\gamma_L(N_{E/L}^M{a_1, \ldots, a_s})). \]
Finally, we observe that

\[ \alpha((\phi_{E/k})_*y) = \alpha((\phi_{L/k})_*((\phi_{E/L})_*y)) = N_{L/k}^M \alpha_L((\phi_{E/L})_*y) \]

\[ = N_{L/k}^M N_{E/L}^M \alpha_E(y) \]

\[ = N_{E/k}^M \alpha_E(y) = N_{E/k}^M \{[P_1], [P_2], a_1, \ldots, a_s\}_{E/E} \]

\[ = \{[P_1], [P_2], a_1, \ldots, a_s\}_{E/k} = g \]

so \(\alpha\) is surjective.

5 The Albanese Kernel

Let \(X\) be a smooth projective variety over a field \(k\) such that \(X(k) \neq \emptyset\). Denote by \(Z_0(X)\) the group of 0-cycles on \(X\), \(Z_0^0(X) \subseteq Z_0(X)\) the subgroup of 0-cycles of degree 0, \(CH_0(X)\) the (Chow) group of 0-cycles on \(X\) modulo rational equivalence and \(A_0(X) \subseteq CH_0(X)\) the subgroup consisting of 0-cycles of degree 0. Fix a point \(P \in X(k)\) and let \(\text{Alb}(X)\) be the Albanese variety of \(X\), with \(f : X \to \text{Alb}(X)\) the associated map sending \(P \mapsto 0\). Then \(Z_0^0(X)\) is generated as a free abelian group by cycles of the form \([Q] - [k(Q) : k][P]\), where \(Q\) is a closed point of \(X\), and hence there is a map \(\alpha_X : Z_0^0(X) \to \text{Alb}(X)(k)\) defined by \(\sum n_P P \mapsto \sum_P n_P N_{k(f(P))/k}(P)\), where the first sum is formal and the second represents the group law on \(\text{Alb}(X)\). It is not hard to see that \(\alpha\) factors through rational equivalence, hence induces a map

\[ \text{alb}_X : A_0(X) \longrightarrow \text{Alb}(X)(k) \]

called the Albanese map. This map is evidently functorial in \(X\). In the case that \(X = A\) is an abelian variety, we obviously have \(\text{Alb}(A) \cong A\).

The main result linking the Albanese kernel to mixed \(K\)-groups is the following:

**Theorem 5.1.** (Raskind-Spiess [RS], Theorem 2.4) Let \(k\) be a field, \(C_1, \ldots, C_n\) smooth projective varieties over \(k\) each containing a \(k\)-rational point. Let \(J_1, \ldots, J_n\) be the respective Jacobians of these curves. Then there is a natural map giving an isomorphism:

\[ \text{Ker} \left( \text{alb}_{C_1 \times_k \ldots \times_k C_n} \right) \xrightarrow{\cong} \bigoplus_{\nu=2}^n \bigoplus_{1 \leq i_1 < \ldots < i_\nu \leq n} K(k; J_{i_1}, \ldots, J_{i_\nu}) \]
6 Behavior with respect to isogenies

In the ensuing discussion, we prove that certain groups related to isogenies of abelian varieties have finite exponent. The following lemma permits us to make finite base extensions as necessary when dealing with mixed $K$-groups.

**Lemma 6.1.** Let $L/k$ be a finite extension of fields, with $X_1, \ldots, X_q$ quasiprojective varieties over $k$, $Y_1, \ldots, Y_r$ projective varieties over $k$, and $G_1, \ldots, G_s$ semi-abelian varieties over $k$. Suppose

$$H_L = K(L; CH_0((X_1)_L), \ldots, CH_0((X_q)_L); A_0((Y_1)_L), \ldots, A_0((Y_r)_L); (G_1)_L, \ldots, (G_s)_L)$$

is a group of finite exponent. Then

$$H = K(k; CH_0(X_1), \ldots, CH_0(X_q); A_0(Y_1), \ldots, A_0(Y_r); G_1, \ldots, G_s)$$

has finite exponent.

**Proof.**

Let $res_{L/k} : H \to H_L$ and $N_{L/k} : H_L \to H$ be (respectively) the covariant and contravariant maps described in Theorem 3.1. Elementary properties of the construction show that $N_{L/k} \circ res_{L/k}$ is multiplication by $[L : k]$. Hence Ker $res_{L/k}$ is a group of finite exponent, which yields the result.

The next few results describe how the Chow groups change under isogeny of abelian varieties:

**Proposition 6.2.** Let $k$ be a field and $f : A \to B$ an isogeny of abelian varieties. Then the kernel and cokernel of the induced map on higher Chow groups $f_* : CH^*(A, \cdot) \to CH^*(B, \cdot)$ are groups of finite exponent.

**Proof.**

Since $f_* \circ f^*$ is multiplication by $n = \deg f = \#\text{Ker } f$, we see that Coker $f_*$ has finite exponent. Note that $n : A \to A$ factors as $g \circ f$ for some isogeny $g : B \to A$. Thus, $n_* = g_* \circ f_*$, so to prove that Ker $f_*$ has finite exponent, it suffices to do the same for $n_*$. Let $g = \dim A$, and suppose $n_* x = 0$ for some $x \in CH^*(A, \cdot)$. Let $\hat{A}$ be the abelian variety dual to $A$, $\ell \in CH^1(A \times_k \hat{A})$ the class of the Poincaré line bundle and $p : A \times_k \hat{A} \to A$, $q : A \times_k \hat{A} \to \hat{A}$ the projections on the factors. Recall that for all $m \neq 0, \pm 1$, $(1 \times m)^* \ell = (m \times 1)^* \ell = m\ell$.  

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Let $N = (2g)!$ and $\mathcal{F}_N : CH^*(A, \cdot) \to CH^*(\hat{A}, \cdot)$, $\hat{\mathcal{F}}_N : CH^*(\hat{A}, \cdot) \to CH^*(A, \cdot)$ denote the “integral” Fourier transforms as defined by Beauville [Be1]. In particular, for $y \in CH^*(A, \cdot)$ and $\hat{y} \in CH^*(\hat{A}, \cdot)$, we have

$$\mathcal{F}_N(y) = q_*(p^*y \cdot \sum_{r=0}^{2g} \frac{N}{r!} \ell^r) \quad \text{and} \quad \hat{\mathcal{F}}_N(\hat{y}) = p_*(q^*\hat{y} \cdot \sum_{r=0}^{2g} \frac{N}{r!} \ell^r).$$

Moreover, $\hat{\mathcal{F}}\mathcal{F} = (-1)^g \sigma^*N^2y$, where $\sigma : A \to A$ is the involution $x \mapsto -x$.

Lemma 6.3. Ker $n^*$ is a group of exponent $n^{2g}$.

Proof.
If $n^*x = 0$, then $n^{2g}x = n_0 n^*x = 0$.

Lemma 6.4.

$$\mathcal{F}_N \circ n_* = n^* \circ \mathcal{F}_N$$

Proof.
Let $y \in CH^*(A, \cdot)$. Then

$$\mathcal{F}_N \circ n_* y = q_*(p^*n_*y \cdot \sum_{r=0}^{2g} \frac{N}{r!} \ell^r)$$

$$= q_*((n \times 1)_*p^*y \cdot \sum_{r=0}^{2g} \frac{N}{r!} \ell^r)$$

$$= q_*((n \times 1)_*(p^*y \cdot (n \times 1)^* \sum_{r=0}^{2g} \frac{N}{r!} \ell^r))$$

$$= q_*((n \times 1)_*(p^*y \cdot \sum_{r=0}^{2g} \frac{N}{r!} n^r \ell^r))$$

$$= q_*(p^*y \cdot \sum_{r=0}^{2g} \frac{N}{r!} n^r \ell^r)$$

On the other hand,

$$n^*\mathcal{F}_N y = n^*q_*(p^*y \cdot \sum_{r=0}^{2g} \frac{N}{r!} \ell^r)$$

$$= q_*(1 \times n)^*(p^*y \cdot \sum_{r=0}^{2g} \frac{N}{r!} \ell^r)$$

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Continuing with the proof of Proposition 6.2, suppose \( n_y = 0 \) for some \( y \in CH^*(A, \cdot) \). Then \( n^* \mathcal{F}_N y = \mathcal{F}_N n_y = 0 \), so by Lemma 6.3, \( n^2g \mathcal{F}_N \hat{y} = 0 \). Thus, \( n^2g \mathcal{F}_N \hat{y} = 0 \), so \( n^2N^2y = 0 \).

The corresponding result for mixed \( K \)-groups is much easier to prove.

**Proposition 6.5.** Let \( X_1, \ldots, X_q, Y_1, \ldots, Y_r, G_2, \ldots, G_s \) be as in Theorem 3.1. Let \( f : A \to B \) be an isogeny of abelian varieties over \( k \). Then the kernel and cokernel of the induced map on mixed \( K \)-groups

\[
  f_* : K(k; CH_0(X_1), \ldots, CH_0(X_q), A_0(Y_1), \ldots, A_0(Y_r), A, G_2, \ldots, G_s) \longrightarrow
  K(k; CH_0(X_1), \ldots, CH_0(X_q), A_0(Y_1), \ldots, A_0(Y_r), B, G_2, \ldots, G_s)
\]

has finite exponent.

**Proof.**

Consider a symbol, \( s = \{x_1, \ldots, x_q, y_1, \ldots, y_r, b, g_2, \ldots, g_s\}_{E/k} \) in the target group. Since the isogeny \( f : A \to B \) is a proper, surjective map, the base extension \( f_E : A_E \to B_E \) is also surjective; hence we may choose \( a \in A_E \) such that \( f_E(a) = b \). Let \( i : E \hookrightarrow E(a) \) be the inclusion of \( E \) in the residue field of \( a \in A_E \), and let \( m = [E(a) : E] \). Then, by a relation of type \textbf{M1},

\[
  ms = \{i^*x_1, \ldots, i^*x_q, i^*y_1, \ldots, i^*y_r, i^*b, i^*g_2, \ldots, i^*g_s\}_{E(a)/k}
  = f_*(\{i^*x_1, \ldots, i^*x_q, i^*y_1, \ldots, i^*y_r, a, i^*g_2, \ldots, i^*g_s\}_{E(a)/k})
\]

However, \( m \leq \deg f_E = \deg f \), so by the above reasoning, we see that the cokernel of \( f_* \) is killed by \( (\deg f)! \).

To see that \( f_* \) is injective, let \( n = \deg f \). Then \( n : A \to A \) factors as \( n = g \circ f \), where \( g : B \to A \) is an isogeny. By functoriality, \( n_* = g_* \circ f_* \) is an endomorphism of \( K(k; CH_0(X_1), \ldots, CH_0(X_q), A_0(Y_1), \ldots, A_0(Y_r), A, G_2, \ldots, G_s) \) which coincides with multiplication by \( n \). Thus, \( \text{Ker } f_* \subseteq \text{Ker } n_* \) is killed by \( n \).
7 Relations in the mixed $K$-group

Let $k$ be a field and $E$ an elliptic curve over $k$. In the following, we describe several relations in mixed $K$-groups. In the interest of keeping the notation simple, the results are stated for the group $K(k; E, E)$, although it is easy to see (by using the "identity map" on the other factors) that the appropriate analogue holds for groups of the form $K(k; CH_0(X_1), \ldots, CH_0(X_q); A_0(Y_1), \ldots, A_0(Y_r); E, E, G_1, \ldots, G_s)$, where the $X_i, Y_j$, and $G_k$ are as in Theorem 3.1.

**Proposition 7.1.** For any finite extension $F/k$ and any $a \in E(L)$,

$$2\{(a, \phi_F(a))\}_{F/k} = 0$$

in the group $K(k; E, E)$.

**Proof.**

Let $E_F = E \times_k F$, and let $\eta_F : \text{Spec } L \to E_F$ be the inclusion of the generic point. Define $\eta : \text{Spec } L \to E$ by $\pi \circ \eta_F$, where $\pi : E_F \to E$ is the base change morphism. Letting $O$ denote the point at infinity on $E$, the divisor $D = (a) + (-a) - 2(O)$ on $E_F$ is principal, so we may pick $h \in L$ such that $\text{div } h = D$.

Now consider the following relation of type $M2$ in $K(k; E, E)$:

$$R_2(L, h, \eta, \phi \circ \eta) = \sum_v \text{ord}_v(h)\{\eta(v), (\phi \circ \eta)(v)\}_{k(v)/k}$$

$$= \{(a, \phi_F(a))\}_{F/k} + \{(-a, \phi_F(-a))\}_{F/k} - 2\{O, \phi_F(O)\}_{F/k}$$

Because the symbols are linear in each factor and $\phi$ is an endomorphism, we have $\{(-a, \phi_F(-a))\}_{F/k} = \{(a, \phi_F(a))\}_{F/k}$ and $\{O, \phi_F(O)\}_{F/k} = 0$, so the above relation implies

$$2\{(a, \phi_F(a))\}_{F/k} = 0 \in K(k; E, E).$$

of type $M2$ in $K(k; E, E)$.

The following also appears in [RM], Lemma 1.7.2.

**Corollary 7.2.** With notation as in Proposition 7.1, let $a, b \in E(F)$. Then

$$\{a, b\}_{F/k} = -\{b, a\}_{F/k}$$

**Proof.**

Preserving the notation in the proof of Proposition 7.1, consider the divisor $D' = (a) + (b) + (-a + b) - 3(O)$. This is a principal divisor on $E_F$, so we may pick $h' \in L$ such that $\text{div } h' = D'$.
Now consider the relation $R_2(L, h', \eta, \eta)$: as above, this implies that \( \{a, a\}_{F/k} + \{b, b\}_{F/k} + \{-(a + b), -(a + b)\}_{F/k} = 0 \), i.e.

\[
\{a, a\}_{F/k} + \{b, b\}_{F/k} + \{a + b, a + b\}_{F/k} = 0, \quad \text{or}
\]

\[
\{a, a\}_{F/k} + \{b, b\}_{F/k} + \{a, a\}_{F/k} + \{b, a\}_{F/k} + \{b, b\}_{F/k} = 0
\]

However, since $2\{a, a\}_{F/k} = 2\{b, b\}_{F/k} = 0$ by Proposition 7.1, so the above implies

\[
\{a, b\}_{F/k} = -\{b, a\}_{F/k}
\]

**Corollary 7.3.** With notation as above, $2\{a, \phi(b)\}_{F/k} = 2\{\phi(a), b\}_{F/k}$

**Proof.**
By Proposition 7.1, $2\{a + b, \phi(a + b)\}_{F/k} = 0$, so $2\{a, \phi(a)\}_{F/k} + 2\{a, \phi(b)\}_{F/k} + 2\{b, \phi(a)\}_{F/k} + 2\{b, \phi(b)\}_{F/k} = 0$. By the same result, the first and last terms are zero, so $2\{a, \phi(b)\}_{F/k} = -2\{b, \phi(a)\}_{F/k}$. By Corollary 7.2, we have $-2\{b, \phi(a)\}_{F/k} = 2\{\phi(a), b\}_{F/k}$.

**Corollary 7.4.** With notation as above, let $\hat{\phi}$ be the isogeny dual to $\phi$, and let $d = \deg \phi$. Then

\[
2\{\phi(a), \hat{\phi}(b)\}_{F/k} = 2d\{a, b\}_{F/k}.
\]

**Proof.**
$2d\{a, b\}_{F/k} = 2\{a, db\}_{F/k} = 2\{a, \phi(\hat{\phi}(b))\}_{F/k} = 2\{\phi(a), \hat{\phi}(b)\}_{F/k}$.

### 8 Supersingular abelian varieties

Let $k$ be a field of positive characteristic and $A$ a supersingular variety over $k$. Recall [O] that when $k$ is algebraically closed, supersingularity of $A$ is characterized by its being isogenous to $E^g$, where $E$ is a supersingular elliptic curve and $g = \dim A$. An elliptic curve $E$ is said to be supersingular if $\text{End}_k(E)$ is an order in a definite quaternion $\mathbb{Q}$-algebra. For a more complete discussion of supersingularity, see [Si], III.9 and V.3.

Our goal is to prove the following result:
**Theorem 8.1.** Let $k$ be a field of positive characteristic, $X_1, \ldots, X_q$, $Y_1, \ldots, Y_r$ as in Theorem 3.1 and $G_1, \ldots, G_s$ semi-abelian varieties over $k$, where $s \geq 0$. Let $E$ be a supersingular elliptic curve over $k$. Then the group

$$K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_q); \mathcal{A}_0(Y_1), \ldots, \mathcal{A}_0(Y_r); E, E, G_1, \ldots, G_s)$$

has finite exponent.

To simplify the proof of the theorem, we note several facts: first, because $\text{End}_k(E)$ is a finitely generated as a $\mathbb{Z}$-algebra, we may pass to a finite extension $L/k$ such that these generators are defined (as morphisms) over $L$, i.e. $\text{End}_L(E) = \text{End}_k(E)$; Lemma 6.1 then guarantees the conclusion for the base field $k$. Hence we may assume that $\text{End}_k(E) = \text{End}_k(E)$.

As in the previous section, we give the proof in the case $q = r = s = 0$, i.e. $\Gamma = K(k; E, E)$. It will be obvious that the same argument works in the general case, but amidst a proliferation of unpleasant notation.

**Proof.**

Let $R = \text{End}_k(E)$ and $\mathcal{K} = R \otimes \mathbb{Q}$. Then $\mathcal{K}$ is a quaternion algebra; as in [Si], p.101, pick elements $\hat{\alpha}, \hat{\beta} \in \mathcal{K}$ such that $\hat{\alpha}^2, \hat{\beta}^2 \in \mathbb{Q}$, $\hat{\alpha}^2 < 0$, $\hat{\beta}^2 < 0$, $\hat{\alpha}\hat{\beta} = -\hat{\beta}\hat{\alpha}$. Pick an integer $D > 0$ such that $D\hat{\alpha}$, $D\hat{\beta} \in R$ and $(D\hat{\alpha})^2$, $(D\hat{\beta})^2 \in \mathbb{Z}$. Now set $\alpha = D\hat{\alpha}$, $\beta = D\hat{\beta}$ and $d_1 = \deg \alpha$, $d_2 = \deg \beta$, $d_3 = \deg \alpha\beta = d_1d_2$.

Then $\alpha\beta + \beta\alpha$ is a torsion element of $R$, but $R = \text{End}_k(E)$ is torsionfree ([Si], III Corollary 7.5), so in fact $\alpha\beta = -\beta\alpha$.

Finally, let $\{a, b\}_{F/k}$ be any element of $K(k; E, E)$. Then, applying Corollaries 7.3 and 7.4 repeatedly, we have

$$2d_3\{a, b\}_{F/k} = 2\{(\alpha\beta)(a), (\alpha\beta)(b)\}_{F/k} = 2\{\alpha\beta(a), \beta\hat{\alpha}(b)\}_{F/k} = 2\{-\beta\alpha(a), \hat{\beta}\hat{\alpha}(b)\}_{F/k} = -2\{-\beta\alpha(a), \hat{\beta}\hat{\alpha}(a)\}_{F/k} = -2d_2\{a, b, \hat{\alpha}(a)\}_{F/k} = -2d_2d_1\{a, b\}_{F/k} = -2d_3\{a, b\}_{F/k}$$

Thus, $4d_3\{a, b\}_{F/k} = 0$ and the theorem is proven.

**Corollary 8.2.** Suppose, in addition to the hypotheses of Theorem 8.1, that the base field $k$ is algebraically closed. Then

$$\Gamma = K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_q); \mathcal{A}_0(Y_1), \ldots, \mathcal{A}_0(Y_r); E, E, G_1, \ldots, G_s) = 0$$

**Proof.**

In this case, the mixed $K$-group is a quotient of the tensor product of $E(k)$ with another group; since $E(k)$ is divisible, so is the mixed $K$-group.
Corollary 8.3. Suppose, in addition to the hypotheses of Theorem 8.1, that the base field $k$ is $C^1$. (cf. [Se], p.161) and that at least one of the $G_i$ is equal to $G_m$.

$$\Gamma = K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_q); A_0(Y_1), \ldots, A_0(Y_r); E, E, G_1, \ldots, G_s) = 0$$

Proof.
This follows from Corollary 3.7 of [A3].

9 Application to higher Chow groups

We now give several applications of Theorem 8.1 to the computation of higher Chow groups of zero-cycles. The following obvious lemma will be invoked repeatedly without mention.

Lemma 9.1. Let $f : A \to B$ be a homomorphism of abelian groups such that $\text{Ker } f$ and $\text{Coker } f$ are of finite exponent. Then $A$ is of finite exponent if and only if $B$ is of finite exponent.

Theorem 9.2. Let $k$ be a field of positive characteristic $E$ a supersingular elliptic curve over $k$. Then for any $r \geq 0$,

$$CH^{r+s}(E^r, s) \cong K_s^M(k) \bigoplus K_s(k; E, G_m)^{\oplus r} \bigoplus F$$

where $F$ is a group of finite exponent.

Proof.
By Theorem 4.2, $CH^{r+s}(E^r, s) \cong K_s(k; \mathcal{CH}_0(E), \ldots, \mathcal{CH}_0(E), G_m)$, which by Theorem 3.4 (and the identification of $E(L)$ with $A_0(E_L)$) gives an isomorphism

$$CH^{r+s}(E^r, s) \cong \bigoplus_{0 \leq \nu \leq r} \bigoplus_{1 \leq i_1 < \ldots < i_r \leq r} K_{\nu,s}(k; E, G_m)$$

Now, the direct summand corresponding to $\nu = 0$ is $K_s(k; G_m) \cong K_s^M(k)$ by [So], Theorem 1.4, and the $r$ direct summands corresponding to $\nu = 1$ are each isomorphic to $K_s(k; E, G_m)$. When $\nu \geq 2$, Theorem 8.1 shows that $K_{\nu,s}(k; E, G_m)$ is a group of finite exponent.

Corollary 9.3. Let $k$ be a field of positive characteristic and $A$ be a supersingular abelian variety of dimension $g$ over $k$. Let $s \geq 0$ be any integer, and $E$ a supersingular
elliptic curve over $k$ such that $A$ is isogenous (over $k$) to $E^g$. Then there is an exact sequence

$$0 \to F_1 \to CH^{g+s}(A, s) \to K_s^M(k) \oplus K_s(k; E; G_m)^{\oplus g} \oplus F \to F_2 \to 0$$

where $F$, $F_1$ and $F_2$ are groups of finite exponent.

**Proof.**

Let $f : A \to E^g$ be a (nonzero) isogeny. By Theorem 6.2, $CH^{g+s}(A, s)$ has finite exponent if and only if the same is true of $CH^{g+s}(E^g, s)$. The result now follows from Theorem 9.2.

**Corollary 9.4.** Let $A$ be an supersingular abelian variety of dimension $g$ over an algebraically closed field of positive characteristic. Let $s \geq 1$ be any integer and $E$ a supersingular elliptic curve together with an isogeny $f : A \to E^g$. Then

$$CH^{g+s}(A, s) \cong K_s^M(k) \bigoplus K_s(k; E; G_m)^{\oplus g}$$

**Proof.**

First suppose $s \geq 1$. By Theorem 4.1, $CH^{g+s}(A, s) \cong K(k; CH_0(A); G_m)$ and $CH^{g+s}(E^g, s) \cong K(k; CH_0(E^g); G_m)$. By [A3], Theorem 3.1, the groups on the right are uniquely divisible, so by Theorem 6.2 the isogeny $f$ induces an isomorphism $f_* : CH^{g+s}(A, s) \cong CH^{g+s}(E^g, s)$. Now by Theorem 3.4, we have:

$$CH^{g+s}(E^g, s) \cong \bigoplus_{0 \leq \nu \leq g} \bigoplus_{1 \leq i_1 < \cdots < i_{\nu} \leq g} K_{\nu,s}(k; E, G_m)$$

When $\nu \geq 2$, $K_{\nu,s}(k; E, G_m)$ is of finite exponent, but, being a quotient of (the divisible group) $E \otimes \cdots \otimes E \otimes k^* \otimes \cdots \otimes k^*$, it is also divisible. Therefore all these terms vanish and the result follows.

The next consequence is a generalization of the Maruyama-Suwa result to supersingular abelian varieties over arbitrary fields.

**Corollary 9.5.** Let $k$ be any field of positive characteristic and $A$ a supersingular variety over $k$. Then $\text{Ker alb} : A_0(A) \to A(k)$ is a group of finite exponent.

**Proof.**

An argument similar to that used in the proof of Lemma 6.1 shows that we may make a finite base extension $L/k$ with no loss of generality. Thus we may assume that there exists an elliptic curve $E$ over $k$ and an isogeny $f : A \to E^g$ (where $g = \dim A$).
By Theorem 6.2, \( f \) induces a map \( f_* : CH_0(A) \to CH_0(E^g) \) which has kernel and cokernel of finite exponent. Using the exact sequence \( 0 \to A_0(X) \to CH_0(X) \xrightarrow{\deg} \mathbb{Z} \), valid for any projective variety \( X \) over \( k \), it is easy to see that \( f_* \) restricts to a map \( f_*^0 : A_0(A) \to A_0(E^g) \) with the same property. Finally, we use the following diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \to & \text{Ker } (\text{alb}_A) & \to & A_0(A) & \xrightarrow{\text{alb}_A} & A(k) & \to & 0 \\
 & & \downarrow f_*^a & & \downarrow f_* & & \downarrow f(k) & & \\
0 & \to & \text{Ker } (\text{alb}_{E^g}) & \to & A_0(E^g) & \xrightarrow{\text{alb}_{E^g}} & E(k)^g & \to & 0
\end{array}
\]

By the previous argument, Ker \( f_* \) and Coker \( f_* \) have finite exponent, while Ker \( f(k) \) is finite; so by the Snake Lemma we conclude that Ker \( f_*^a \) and Coker \( f_*^a \) are also of finite exponent.

Now Theorem 5.1 implies that Ker \( \text{alb}_{E^g} \) is a group of finite exponent, so the preceding reasoning shows that the same is true of Ker \( \text{alb}_A \).

Finally, we show how to recover the result of Maruyama and Suwa:

**Corollary 9.6.** Suppose, in addition to the hypotheses of Corollary 9.5, that \( k \) is algebraically closed and of positive characteristic. Then Ker \( \text{alb}_A \) = 0.

**Proof.**

Corollary 9.5 tells us that this group is of finite exponent, but we also know [Bl1] that \( \text{alb}_A \) is equal to the image of the “Pontryagin product” map \( A_0(A) \otimes A_0(A) \to A_0(A) \) defined on generators by \( ([a] - [0]) \otimes ([b] - [0]) \mapsto [a + b] - [a] - [b] + [0] \) and extending in the appropriate manner by linearity. Since \( A_0(A) \) is divisible ([Bl2], Lemma 1.4), this shows that \( \text{alb}_A \) is divisible, hence zero.
References


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