The Representation Number of Some Sparse Graphs

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Abstract

We study the representation number for some special sparse graphs. For graphs with a single edge and for complete binary trees we give an exact formula, and for hypercubes we improve the known lower bound. We also study the prime factorization of the representation number of graphs with one edge.

1 Introduction

A finite graph $G$ is representable modulo $r$ if there exists an injective map $f : V(G) \to \{0, 1, \ldots, r - 1\}$ such that for all vertex pairs, $uv \in E(G)$ if and only if $\gcd(f(u) - f(v), r) = 1$. This is equivalent to requiring that there exist an injective map $f : V(G) \to \mathbb{Z}_r$ such that for all $u, v \in V(G)$, $f(u) - f(v)$ is a unit of (the ring) $\mathbb{Z}_r$ if and only if $uv \in E(G)$. The representation number of $G$, denoted $\text{rep}(G)$, is the smallest positive integer $r$ modulo which $G$ is representable. Representation numbers first appeared in [3] and were used by Erdős and Evans to give a simpler proof of a result of Lindner et al. [7] that any finite graph can be realized as an orthogonal Latin square graph – that is, for any graph, there is an assignment of Latin squares (of the same order) to the vertices in such a way that vertices are adjacent if and only if the associated Latin squares are orthogonal. Representation numbers have been determined for complete graphs [5], edgeless graphs [5], and stars [1] ; there are also bounds and partial results for representation numbers of complete multipartite graphs [2], disjoint unions of complete graphs [4], and various other graph families ([6], [10]).
Representations modulo \( r \) are closely related to so-called \textit{product representations} of graphs. A product representation of a graph \( G \) is a labeling of its vertices by integer \( k \)-tuples in such a way that two vertices are adjacent if and only if their labels differ in all coordinates; the \textit{product dimension} of \( G \), denoted \( \text{pdim} \ G \), is the least positive integer \( k \) for which this is possible. Now if \( r \) is a squarefree integer with prime factorization \( p_1 \cdots p_s \), the Chinese Remainder Theorem gives an isomorphism \( \mathbb{Z}_r \cong \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s} \). This provides a natural method to convert mod \( r \) representations of a graph into product representations and vice versa. If \( G \) is a reduced graph (i.e. no two vertices have the same open neighborhood), \( r = \text{rep}(G) \) will always be squarefree [5]. In this case, it is easy to see that the number of prime factors in \( \text{rep}(G) \) must be at least \( \text{pdim} \ G \); however, it is not known whether these two quantities are equal. Even when \( G \) is not reduced, product representations are often used to establish upper bounds for the representation number: a general upper bound (for all graphs) is proved by this method [9], as is an upper bound for the representation number of the hypercube [10]. In Section 4 of the present work, we use this same method when studying complete binary trees.

The purpose of this article is to study the representation number of particular families of sparse graphs. We first consider the graph \( S_n = K_2 + nK_1 \). Representations of \( S_n \) were first studied in [5] and an upper bound of \( 6n \) established for \( \text{rep}(S_n) \). Narayan and Urick [10] improved this bound and conjectured that it gave the true value of \( \text{rep}(S_n) \). We prove their conjecture for \( n \) sufficiently large using bounds on arithmetic functions from [11] and show that \( \text{rep}(S_n) \) is close to \( 2n \). Since the Narayan-Urick formula involves functions which are difficult to compute, we adapt the techniques developed in [1] and [2] to give a partial description of the prime factors of \( \text{rep}(S_n) \), reminiscent of the description of \( \text{rep}(K_m + nK_1) \) in [6, Section 5] for small values of \( n \). In Section 4, we compute the representation number of complete binary trees, giving an exact formula in all cases except for the tree on 15 vertices. In Section 5, by examining more closely the construction of Narayan and Urick [10] and establishing an easy lower bound, we improve known results about the representation number of the hypercube.

Throughout this paper, various sums and products indexed by a set of prime numbers appear: in all such formulas \( p \) or \( q \) indicates a prime. The notation \( \gamma \) is reserved for the Euler-Mascheroni constant and \( p_1, p_2, \ldots \) for the prime numbers, with \( p_1 < p_2 < \cdots \). We define the \textit{primorial} \( \varphi_n \) to be \( \prod_{i=1}^{n} p_i \). The \textit{radical} of a nonzero integer \( n \), denoted \( \text{rad} \ n \), is the product of the distinct primes dividing it.
2 Preliminaries

In this section we collect several definitions and tools which will be used throughout the article.

Given a graph $G$, define (following [5]) an equivalence relation on $\sim$ on $V(G)$ by declaring two vertices to be equivalent if they share the same open neighborhood in $G$. Letting $[u]$ denote the equivalence class of $u \in V(G)$, define the reduction of $G$, denoted $\hat{G}$, by

$$V(\hat{G}) = \{[u] : u \in V(G)\}$$

and

$$[u][v] \in E(\hat{G}) \iff uv \in E(G)$$

It is immediate that the above description of $E(\hat{G})$ is well-defined. A graph is called reduced if no two distinct vertices share the same open neighborhood. The main result of relevance to us is the following:

**Proposition 2.1.** [5, Lemma 2.4] Suppose $p_1, \ldots, p_s$ are distinct primes and $e_1, \ldots, e_s$ positive integers. If $G$ is representable modulo $p_1^{e_1} \ldots p_s^{e_s}$, then $\hat{G}$ is representable modulo $p_1 \ldots p_s$. In particular, if $G$ is reduced then $\text{rep}(G)$ is squarefree.

For reduced graphs, the product dimension can be used to give a lower bound on the representation number.

**Proposition 2.2.** [6, Theorem 2.11] Let $G$ be a reduced graph. Then

$$\text{rep}(G) \geq p_\ell p_{\ell + 1} \ldots p_{\ell + m - 1}$$

where $m = \text{pdim } G$ and $\ell$ is the smallest integer satisfying $p_\ell \geq \chi(G)$.

We also record the following result of Lóvasz et al. as a helpful tool in establishing a lower bound for the product dimension.

**Lemma 2.3.** [8] Let $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$ be two lists of vertices in a graph $G$. If $u_i$ is adjacent to $v_j$ when $i = j$ and $u_i$ is not adjacent to $v_j$ when $i < j$, then $\text{pdim } G \geq \lceil \log_2 r \rceil$.

Finally, we will need some explicit estimates on the values of certain arithmetic functions from the well-known paper of Rosser and Schoenfeld [11].
Lemma 2.4.  

- [11, Theorem 4 and Corollary] For $x > 1$,
  \[ e^{x(1 - \frac{1}{\log x})} < \prod_{q \leq x} q < e^{x(1 + \frac{1}{\log x})} \]

- [11, Theorem 7 and Corollary] For $x > 1$,
  \[ \frac{e^{-\gamma}}{\log x} \left( 1 - \frac{1}{\log^2 x} \right) < \prod_{q \leq x} \left( 1 - \frac{1}{q} \right) < \frac{e^{-\gamma}}{\log x} \left( 1 + \frac{1}{2\log^2 x} \right) \]

- [11, p. 72] For $n \geq 3$,
  \[ \frac{n}{\phi(n)} < e^{\gamma \log \log n + 2.50637/\log \log n} \]
  In particular, if $n \geq 12$,
  \[ \frac{n}{\phi(n)} < 5 \log \log n \]

3 Graphs with a single edge

In this section we study and determine the representation number of the graph $S_n = K_2 + nK_1$, which has $n + 2$ vertices and a single edge. To simplify notation, define

\[ M_n = \min\{2^k m : k \geq 1, m \geq 3 \text{ is odd, and } 2^{k-1}(m - \phi(m)) \geq n\}. \]

Narayan and Urick proved that $M_n$ is an upper bound for $\text{rep}(S_n)$; we include a proof here in the interest of completeness of presentation.

Proposition 3.1. [10, Corollary 1]

\[ \text{rep}(S_n) \leq M_n. \]

Proof.

Given $k$ and $m$ as in the definition of $M_n$, define a labeling modulo $2^km$ by assigning to 0 and 1 to the adjacent vertices of $S_n$ and assigning to the isolated vertices elements from the set

\[ T = \{ l : 0 \leq l \leq r-1, \ l \equiv i (\text{mod } 2^k), \ l \equiv j (\text{mod } m), \ i \text{ is even, and } \gcd(j, m) \neq 1 \}. \]

This is possible because there are $2^{k-1}$ choices for $i$ and $m - \phi(m)$ choices for $j$; hence the Chinese Remainder Theorem implies $|T| = 2^{k-1}(m - \phi(m))$. Thus, there are at least $n$ elements of $T$ which can be used to label the isolated vertices. 

Our goal is to prove that when $n$ is sufficiently large, $\text{rep}(S_n) = M_n$. 

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**Lemma 3.2.** For every \( \varepsilon > 0 \) there exists \( n_0 \) such that for \( n > n_0 \), \( 2n \leq \text{rep}(S_n) < 2(1+\varepsilon)n \). Furthermore, for sufficiently large \( n \), \( 2 \) divides \( \text{rep}(S_n) \).

**Proof.**

Since \( K_n \) is an induced subgraph of \( S_n \), \( \text{rep}(S_n) \geq \text{rep}(K_n) = 2n \) by [5, Example 1.1]. For \( x > 1 \), Lemma 2.4 implies \( \beta_x = \prod_{p \leq x} (1 - \frac{1}{p}) \leq e^{-\gamma(1 + \frac{1}{2\log^2 x})} \). Now fix \( \varepsilon > 0 \). By choosing \( x \) such that \( \log x > \frac{8}{\varepsilon}e^{-\gamma(1 + \frac{1}{2\log^2 x})} \), we may ensure that \( \delta = (1 - 2\beta_x)^{-1} < (1 + \varepsilon/4) \). Now let \( s_x = \prod_{3 \leq p \leq x} p \) and note that by Lemma 2.4, \( \frac{1}{2} e^{x(1 - \frac{1}{\log x})} \leq s_x \leq \frac{1}{2} e^{x(1 + \frac{1}{\log x})} \). For \( n > \frac{4e^{x(1 + \frac{1}{\log x})}}{\varepsilon} \), there is a number of the form \( ks_x \), with \( k \) odd, in the interval \( (\delta n, (1+\varepsilon/2)n) \). Now let \( r = 2ks_x \). Next, \( ks_x - \phi(ks_x) = ks_x (1 - \prod_{p|ks_x, p \geq 3} (1 - \frac{1}{p}) ) \geq ks_x (1 - 2\prod_{p \leq x} (1 - \frac{1}{p}) ) \geq n \); so Proposition 3.1 implies \( \text{rep}(S_n) \leq 2ks_x < 2(1+\varepsilon)n \).

To prove the second assertion, note that [1, Lemma 2.6] implies that \( p_0 \), the smallest prime dividing \( \text{rep}(S_n) \), satisfies \( p_0 \leq \frac{\text{rep}(S_n)}{n+1} \). Thus, if \( n \) is sufficiently large to ensure \( \text{rep}(S_n) < 3n \), we have \( p_0 < 3 \). Thus \( p_0 = 2 \), as desired. \( \square \)

We are now in a position to prove our main result.

**Theorem 3.3.** For sufficiently large \( n \),

\[ \text{rep}(S_n) = M_n. \]

**Proof.**

Let \( r = \text{rep}(S_n) \) and let \( x \) and \( y \) denote adjacent vertices of \( S_n \). We know from Lemma 3.2 that \( r = 2^km \) for some \( k \geq 1 \) and some odd integer \( m \geq 1 \). If \( m = 1 \), then \( r = 2^k \), which is impossible since the only graphs representable modulo a prime power are complete multipartite graphs. Now fix a labeling of \( S_n \) modulo \( r \); for convenience, we consider labels as elements of \( \mathbb{Z}_{2^k} \times \mathbb{Z}_m \) and denote by \( \pi : \mathbb{Z}_{2^k} \rightarrow \mathbb{Z}_2 \) the natural quotient map. Without loss of generality (cf. [1, Lemma 2.3]) we may assume that \( x \) and \( y \) are labeled \((0,0)\) and \((1,1)\), respectively. Let \( A \) be the set of vertices in \( S_n \) distinct from \( x \) which have labels of the form \((a,i)\) where \( \pi(a) = 0 \) and \( B \) the set of vertices distinct from \( y \) having labels of the form \((a,j)\) where \( \pi(a) = 1 \). Observe that
$A \cup B$ is an independent set in $S_n$. Now for any vertex of $B$ having label $(a,i)$, note that there is no vertex of $A$ having label $(a+1,i+1)$, for then the difference between the two labels would be a unit, forcing the existence of an edge between a vertex of $A$ and a vertex of $B$. Hence we may relabel all vertices of $B$, replacing the label $(a,i)$ with $(a+1,i+1)$. It is easy to check that this new labeling gives a representation modulo $r$ in which every vertex in $A \cup B$ has a label of the form $(a,i)$ with $\pi(a) = 0$. However, none of these vertices are adjacent to $y$, which is labeled $(1,1)$, and since $a-1 \in \mathbb{Z}_{2^k}^*$, it must be the case that $i-1 \notin \mathbb{Z}_m^*$. Hence there are $2^{k-1}$ choices for $a$ and $m - \phi(m)$ choices for $i$. This forces $n = |A \cup B| \leq 2^{k-1}(m - \phi(m))$, as desired.

Calculations of the prime factorization of $\text{rep}(S_n)$ suggest a pattern: most of the exponents on the prime factors of $\text{rep}(S_n)$ are equal to 1, and $\text{rep}(S_n)$ seems to divisible by a primorial which grows as a function of $n$. In Proposition 3.5 and Corollary 3.7 we make these observations rigorous; our results are similar in form to those for the representation number of $K_m + nK_1$ (for small $n$) from [6, Section 5]. The strategy of the proof, however, is quite different from that of [6] but rather similar to that of [1] and [2]: we assume that the representation number is not of the claimed form, and then prove that it is forced to exceed some known upper bound.

For the balance of this section, we reserve the notation $r$ for $\text{rep}(S_n)$; for sufficiently large $n$ we also define $k$ and $s$ by $r = 2^k s$, where $k \geq 1$ and $s$ is odd. We begin with a result that says essentially that for large enough $n$ we may assume that $\text{rep}(S_n)$ is divisible by some sufficiently large prime.

**Lemma 3.4.** For every $x > 0$ there exists $n_0$ such that for $n > n_0$, $\text{rep}(S_n)$ is divisible by some prime $q \geq x$.

**Proof.**
If $r$ is not divisible by any prime $q \geq x$, then

$$n \leq 2^{k-1}(s - \phi(s)) \leq 2^{k-1}s(1 - \prod_{p \leq x}(1 - \frac{1}{p})) \leq 2^{k-1}s(1 - \frac{e^{-\gamma}}{2 \log x}) = \frac{r}{2}(1 - \frac{e^{-\gamma}}{2 \log x})$$

by Lemma 2.4. Thus,

$$r \geq \frac{2n}{1 - \frac{e^{-\gamma}}{2 \log x}}.$$ 

Since $x$ is fixed, Lemma 3.2 yields a contradiction for sufficiently large $n$. 

We may now prove that $s$ is close to being squarefree.

**Proposition 3.5.** For sufficiently large $n$, $\text{rad } s > \frac{s}{40(\log s)^2 \log \log s}$. 

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By Lemma 3.4, we may assume $s \geq 12$. Suppose $t = \text{rad} s$ and let $u = s/t$; observe that $u$ is odd. Let $p$ be the smallest prime number not dividing $r$; we claim $t \geq \frac{s}{10p^2 \log \log s}$. If $u \leq 3p$, then $t \geq \frac{s}{3p} > \frac{s}{10p^2 \log \log s}$. Otherwise, write $u = \ell p + c$ for some odd integer $\ell$ and some integer $c$, $0 < c < 2p$. Now let $s' = t\ell p$ and $r' = 2^ks'$. Since $r' < r = \text{rep}(S_n)$, it follows from Theorem 3.3 that $s' - \phi(s') < s - \phi(s)$ or equivalently $t\ell p - \phi(t\ell p) < tu - \phi(tu)$, which reduces to

$$\phi(tu) - \phi(t(u-c)) < tc.$$  (1)

However, $p$ divides $u - c$ and every prime dividing $u$ also divides $t$, so

$$\phi(t(u-c)) = t(u-c) \prod_{q|t(u-c)} \left(1 - \frac{1}{q}\right) \leq \frac{u-c}{u} (1 - \frac{1}{p}) \phi(tu)$$

which, after substituting into (1), simplifies to

$$\frac{\phi(s)}{s} < \frac{c}{u - (u-c) \frac{p-1}{p}}.$$

Now $s \geq 12$, so combining the above with the estimate $\frac{\phi(s)}{s} > \frac{1}{5 \log \log s}$ of Lemma 2.4, we have $\frac{1}{p} + \frac{c(p-1)}{s_p^p} < 5c \log \log s$. Hence $u < 5cp \log \log s < 10p^2 \log \log s$ and so $t = \text{rad} s > \frac{s}{10p^2 \log \log s}$ in this case also.

From Lemma 2.4 we have $e^{p(1-\frac{1}{\log p})} < \prod_{q \leq p} q \leq 2ps$; thus $p < 2 \log s$ and hence

$$\text{rad } s > \frac{s}{40(\log s)^2 \log \log s}.$$

\[\square\]

**Lemma 3.6.** Let $p$ the smallest prime that does not divide $s$ and $p'$ the largest prime divisor of $s$. For sufficiently large $n$, $p > \frac{1}{4} \sqrt{\frac{p'}{\log p'}}$.

**Proof.**

Observe that if $p' < 3p$, then by Lemma 3.4, $p > \frac{p'}{3} > \frac{1}{4} \sqrt{\frac{p'}{\log p'}}$; so assume henceforth $p' > 3p$. Define $a$ by $s = p'^a m$ where $\gcd(m, p') = 1$ and write $p'^a = \ell p + c$ with $\ell$ odd, $0 < c < 2p$. Next, define $s' = s\ell \frac{p}{p'^a}$. Since $2^k s' < 2^k s = r$, Proposition 3.3 implies $s' - \phi(s') < s - \phi(s)$. However, $\phi(s') \leq \frac{p-1}{p} \phi(s)$; thus we have:

$$\phi(s)(1 - \frac{p'(p-1)}{p(p'-1)}) < s(1 - \frac{\ell p}{p'^a}) = \frac{cs}{p'^a} < \frac{2ps}{p'}$$
which in turn implies \( p' < p(2p - \frac{s}{\phi(s)} + 1) \). From Lemma 2.4, we have \( \text{rad } s \leq e^{p'(1 + \frac{1}{2\log p'})} \); so in fact \( \log \text{rad } s < \frac{3}{2} p' \). Moreover, Proposition 3.5 gives the estimate \( \log s < 2 \log \text{rad } s \). Again by Lemma 2.4, \( \frac{s}{\phi(s)} < 5 \log \log s < 5 \log (3p') < 6 \log p' \); thus,
\[
p' < p(6p \log p' + 1) < 16p^2 \log p'.
\]

Remark.
We used very rough estimates to obtain the constants in Proposition 3.5 and Lemma 3.6 in the interest of keeping the arithmetic simple. Their precise values are not important for the proof of our main result on the prime factors of \( \text{rep}(S_n) \).

**Corollary 3.7.** Fix \( \delta > 0 \). For sufficiently large \( n \), \( \text{rep}(S_n) \) is divisible by all primes less than \( (\log n)^{\frac{1}{16}(1-\delta)} \).

**Proof.**
Let \( p \) and \( p' \) be as in Lemma 3.6. By Proposition 3.3, \( \frac{r}{2}(1 - \frac{\phi(s)}{s}) \geq n \). Also, by Lemma 2.4,
\[
\frac{\phi(s)}{s} = \prod_{q|s}(1 - \frac{1}{q}) \geq \prod_{3 \leq q \leq p'}(1 - \frac{1}{q}) \geq \frac{e^{-\gamma} \eta}{\log p'},
\]
where \( \eta = 1 - \frac{1}{\log^2 p'} \); so in fact \( \frac{r}{2}(1 - \frac{e^{-\gamma} \eta}{\log p'}) \geq n \) and hence
\[
r \geq 2n(1 + \frac{\eta e^{-\gamma}}{\log p' - \eta e^{-\gamma}}).
\]

Let \( \varepsilon = \frac{\eta e^{-\gamma}}{\log p' - \eta e^{-\gamma}} \). From the argument in Lemma 3.2 we see that \( r < 2(1 + \varepsilon)n \) if we choose \( n > \frac{4e^{x(1 + \frac{1}{2\log x})}}{\varepsilon} \), where
\[
x > \exp(8e^{-\gamma}(1 + \frac{1}{2\log^2 x})(1 + \frac{1}{\varepsilon})) = \exp(8\frac{\log p'}{\eta}(1 + \frac{1}{2\log^2 x})) = p'^{\frac{8}{7}(1 + \frac{1}{2\log^2 x})}.
\]

Lemma 3.4 implies that for sufficiently large \( n \), \( p' \) may be made arbitrarily large; hence we may make \( \eta \) arbitrarily close to 1. Hence in order for (2) to hold, we must have \( n < \exp(p'^{8}(1 + \delta)) \) or \( p' > (\log n)^{\frac{1}{16}(1 + \delta)} \). By Lemma 3.6, we may choose \( n \) sufficiently large to guarantee \( p > p'^{\frac{1}{2}(1-\delta^2)} \), so \( p > (\log n)^{\frac{1}{16}(1-\delta)} \), as desired. \( \square \)
4 The complete binary tree

In this section we compute the representation number of the complete binary tree $B_n$ with $2^n - 1$ vertices. Define the level of a vertex $v$, denoted $\ell(v)$, as follows: the root vertex has level 0, and if $v$ is a vertex of level $i$, its heretofore unlabeled neighbors each have level $i + 1$.

It is easy to check that $\text{rep}(B_1) = 1$, $\text{rep}(B_2) = 4 = 2^2$, and $\text{rep}(B_3) = 12 = 2^2 \cdot 3$.

**Theorem 4.1.** For all $n \geq 5$, $\text{rep}(B_n) = \varphi_n$.

**Proof.**

Note that when $n \geq 2$, $B_n$ is not a reduced graph; indeed, there is a unique partition of the leaves of $B_n$ into pairs such that two distinct vertices of $B_n$ share the same open neighborhood if and only if they constitute one of these pairs. Thus, $\hat{B}_n$ is isomorphic to the subgraph of $B_n$ obtained by deleting one leaf of $B_n$ from each such pair. Now suppose $n \geq 5$ and let $a_1, a_3, \ldots, a_{2^n-1}$ denote the leaves of $\hat{B}_n$. For $i$, $1 \leq i \leq 2^{n-2}$, let $b_{2i-1}$ denote the parent of $a_{2i-1}$; for $j$, $1 \leq j \leq 2^{n-3}$, let $c_{4j+2}$ denote the common parent of $b_{4j-3}$ and $b_{4j-1}$. Finally, let $d$ denote the parent of $c_2$ and $e$ the parent of $d$. Now for $j$, $1 \leq j \leq 2^{n-3}$, define

$$ u_{4j-3} = a_{4j-3}, \quad v_{4j-3} = u_{4j-2} = b_{4j-3}, \quad v_{4j-2} = u_{4j-1} = c_{4j-2}, \quad v_{4j-1} = b_{4j-1} = u_{4j}, \quad v_{4j} = a_{4j-1} $$

and set $u_{2^n-1+1} = d$, $v_{2^n-1+1} = e$.

For $1 \leq i \leq 2^{n-1} + 1$, the vertices $u_i$ and $v_i$ satisfy the hypotheses of Lemma 2.3, so $\text{pdim } \hat{B}_n \geq n$. By Proposition 2.2, $\text{rep}(\hat{B}_n) \geq \varphi_n$, and hence by Proposition 2.1, $\text{rep}(B_n) \geq \varphi_n$.

For the upper bound, we construct a labeling (which is also a product representation) inductively. The labeling for $n = 5$ is given in Figure 2; for convenience of notation, we write $abcd$ to represent the label $(a, b, c, d) \in \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7$.

Suppose a representation $g_{n-1} : V(B_{n-1}) \to \mathbb{Z}_2 \times \mathbb{Z}_3 \times \ldots \times \mathbb{Z}_{p_{n-1}}$ is given. Let $r$ denote the root vertex of $B_n$, $x$ its left child, $y$ its right child, and let $B_n(x)$ and $B_n(y)$ denote the subtrees of $B_n$ rooted at $x$ and $y$, respectively. We will construct a labeling $g_n : V(B_n) \to \mathbb{Z}_2 \times \mathbb{Z}_3 \times \ldots \mathbb{Z}_{p_n}$; we begin by defining $g_n$ on the subtrees $B_n(x)$ and $B_n(y)$. Fix bijections $h_1 : B_n(x) \to B_{n-1}$ and $h_2 : B_n(y) \to B_{n-1}$. If $v \in B_n(x)$ and $g_{n-1}(h_1(v)) = (a_1, \ldots, a_{n-1})$, define $g_n(v) = (a_1, \ldots, a_{n-1}, 1)$ if $a_1 = 0$ or $g(v) = (a_1, \ldots, a_{n-1}, 0)$ if $a_1 = 1$. Similarly, if $v \in B_n(y)$ and $g_{n-1}(h_2(v)) = (a_1, \ldots, a_{n-1})$, define $g_n(v) = (a_1, \ldots, a_{n-1}, 0)$ if $a_1 = 0$ or $g_n(v) = (a_1, \ldots, a_{n-1}, 1)$ if $a_1 = 1$. 

Figure 1: Labeling of $B_4$

Figure 2: Labeling of $B_5$
The definition of $g_n$ on the root vertex $r$ is more complicated. First define $g_6(r) = (1, 0, 2, 0, 0, 2)$ and $g_7(r) = (0, 2, 1, 3, 3, 1, 2)$. We now give the definition of $g_n(r) = (c_1, \ldots, c_r)$ when $n \geq 8$.

If $n$ is even, set

$$c_1 = 1, \ c_2 = 1, \ c_3 = 2, \ c_4 = 0, \ c_5 = 0, \ c_{n-2} = 0, \ c_{n-1} = 1, \ c_n = 2.$$ 

Now if $m$ is odd and $7 \leq m \leq n-3$, set $c_m = 2$. If $m$ is even and $6 \leq m \leq n-4$, set $c_m = 0$.

If $n$ is odd, set

$$c_1 = 0, \ c_2 = 2, \ c_3 = 1, \ c_4 = 3, \ c_5 = 3, \ c_{n-2} = 0, \ c_{n-1} = 1, \ c_n = 2.$$ 

Now if $m$ is even and $6 \leq m \leq n-3$, set $c_m = 2$. If $m$ is odd and $7 \leq m \leq n-4$, set $c_m = 0$.

It remains to verify that $g_n$ is a representation of $B_n$ modulo $\varphi_n$. To this end, select distinct vertices $u, v \in B_n$ and let $g_n(u) = (a_1, \ldots, a_n)$, $g_n(v) = (b_1, \ldots, b_n)$. We divide the argument into several cases.

First, suppose both vertices both belong to $B_n(x)$. Suppose that both vertices belong to $B_n(x)$. If they are adjacent, then since $g_n$ (restricted to $B_n(x)$) is defined in terms of $g_{n-1}$, which is a representation of $B_{n-1}$, we have $a_i \neq b_i$ for $1 \leq i \leq n-1$; moreover, since $a_1 \neq b_1$, we must have $a_n \neq b_n$. If $u$ and $v$ are not adjacent, then $a_i = b_i$ for some $i$, $1 \leq i \leq n-1$. The same reasoning applies if both vertices belong to $B_n(y)$.

Next, suppose (without loss of generality) that $u$ belongs to $B_n(x)$ and $v$ belongs to $B_n(y)$. Because $u$ is not adjacent to $v$ in $B_n$, we need to show that $a_i = b_i$ for some $i$, $1 \leq i \leq n$. If $a_1 \neq b_1$, then either $a_1 = 0$ and $b_1 = 1$ or $a_1 = 1$ and $b_1 = 0$. In the first case, the construction of $g_n$ implies that $a_n = b_n = 1$ and in the second that $a_n = b_n = 0$.

Last, suppose $u$ is the root vertex, $v \in B_n(x)$, and $n \geq 6$. (If $v \in B_n(y)$, a similar argument applies.) We divide this case into two subcases: $v = x$ and $v \neq x$. If $v = x$ and $n = 6$, then each coordinate of $g_n(u) = (1, 0, 2, 0, 0, 2)$ differs from $g_n(v) = (0, 1, 0, 1, 1, 0)$. Suppose $n \geq 7$ is odd; then $(a_1, \ldots, a_5) = (0, 2, 1, 3, 3)$. Moreover, if $6 \leq m \leq n-3$, then $a_m = 2$ if and only if $m$ is even, and furthermore we have $a_{n-2} = 0$, $a_{n-1} = 1$, $a_n = 2$. On the other hand, $(b_1, \ldots, b_5) = (1, 1, 2, 0, 0)$. If $6 \leq m \leq n-4$, then $b_m = 2$ if and only if $m$ is odd, and furthermore we have $b_{n-3} = 0$, $b_{n-2} = 1$, $b_{n-1} = 2$, and $b_n = 0$ (the latter since $v \in B_n(x)$). Therefore, $a_i \neq b_i$ for all $i$. If $n \geq 8$ is even, a similar argument can be made.
Finally, if \( v \neq x \), then we have \( \ell(v) \leq n - 3 \). Thus \( u = r \) is not adjacent to \( v \) and we must show that \( a_i = b_i \) for some \( i, 1 \leq i \leq n \). If \( n = 6 \), it is easy to check that \( g_n(u) = (1, 0, 2, 0, 0, 2) \) has at least one coordinate in common with any vertex of level 1 or 3. Next suppose that \( n \geq 7 \) is odd. If \( \ell(v) \) is odd, then \( a_1 = b_1 \), so suppose \( \ell(v) \) is even. If \( \ell(v) = 4 \), then \( a_3 = b_3 = 1 \) regardless of the choice of \( v \). If \( \ell(v) \geq 6 \), \( b_{\ell(v)} = 2 \), and since \( \ell(v) \) is even and \( n \) is odd, \( a_{\ell(v)} = 2 \) by construction. Finally, suppose \( n \geq 8 \) is even. If \( \ell(v) \) is even, then \( a_1 = b_1 \), so suppose \( \ell(v) \) is odd. If \( \ell(v) = 5 \), we have \( a_2 = b_2 = 1 \) for any choice of \( v \); if \( \ell(v) = 3 \), \( a_3 = b_3 = 2 \) for any choice of \( v \), and if \( \ell(v) = 1 \), then either \( b_4 = 0 \) or \( b_5 = 0 \); so \( a_k = b_k \) for some \( k, 4 \leq k \leq 5 \).

\[ \square \]

\textbf{Remark.}

The labeling in Figure 1 shows that \( B_4 \) is representable modulo \( \varphi_4 \). However, the methods used in the proof of Theorem 4.1 only guarantee a lower bound of 3 for \( \text{pdim} \hat{B}_4 \). Hence, we may only conclude \( 30 \leq \text{rep}(B_4) \leq 210 \). It is easy to show \( \text{pdim} B_4 = 4 \) and it seems likely that \( \text{pdim} \hat{B}_4 = 4 \) also, although we do not know how to prove this at present.

\section{The hypercube}

In this section, we consider the problem of determining the representation number of the hypercube \( Q_n \); this is the graph whose vertices are 0,1-strings of length \( n \), with two strings adjacent if and only if they differ in exactly one bit. Note that \( Q_n \) is a reduced graph, so by Proposition 2.1, \( \text{rep}(Q_n) \) is squarefree. Narayan and Urick studied this problem; by considering product representations, they established the following bounds:

\textbf{Theorem 5.1.} \cite[Theorem 5 and Corollary 4]{10}

\[ \text{For } n \geq 3, \text{rep}(Q_n) \leq \varphi_n/3 \text{ and pdim } Q_n \leq n - 1. \]

The proof of Theorem 5.1 is inductive: given a representation of \( Q_n \) modulo \( \varphi_n/3 \), the authors explicitly construct a representation of \( Q_{n+1} \) modulo \( \varphi_{n+1}/3 \). An important aspect of their construction is that for each odd prime \( p \) with \( 5 \leq p \leq p_n \), every label used is congruent modulo \( p \) to one of four residue classes. Thus, their proof actually shows the following:

\textbf{Lemma 5.2.} If \( n \geq 3 \) and \( Q_n \) is representable modulo \( r \), then \( Q_{n+1} \) is representable modulo \( rp \), where \( p \) is any prime at least 5 that does not divide \( r \).
Towards a lower bound for \( \text{rep}(Q_n) \), we record the following result which is surely known to experts. In the interest of completeness, we provide a proof.

**Proposition 5.3.** For \( n \geq 3 \), \( \text{pdim } Q_n = n - 1 \) and \( \text{rep}(Q_n) \geq \varphi_{n-1} \).

**Proof.**
From Theorem 5.1 we have \( \text{pdim } Q_n \leq n - 1 \). For the other inequality, we use Lemma 2.3. Let \( r = 2^{n-1} \) and let \( x_1, \ldots, x_r \) be any ordering of the 0,1-strings of length \( n \) beginning with 0. Now for \( i, 1 \leq i \leq r \), construct \( y_i \) from \( x_i \) by changing the first bit to 1. It is clear from the construction that \( x_i \) is adjacent to \( y_j \) in \( Q_n \) if and only if \( i = j \). Thus, \( \text{pdim } Q_n \geq \log_2 r = n - 1 \). The second statement follows from Proposition 2.2.

We may now narrow the value of \( \text{rep}(Q_n) \) down to two possibilities.

**Corollary 5.4.** If there exists \( n_0 \geq 3 \) such that \( \text{rep}(Q_{n_0}) = \varphi_{n_0-1} \), then \( \text{rep}(Q_n) = \varphi_{n-1} \) for all \( n \geq n_0 \); otherwise, \( \text{rep}(Q_n) = \varphi_n/3 \) for all \( n \geq 3 \).

**Proof.**
First note that \( \text{rep}(Q_n) \) must be divisible by 2: if this were not the case, then Proposition 5.3 would imply \( \text{rep}(Q_n) \geq \varphi_n/2 \), contradicting Theorem 5.1. If \( \text{rep}(Q_n) \) is not divisible by 3, then Proposition 5.3 forces \( \text{rep}(Q_n) \geq \varphi_n/3 \), which when combined with Theorem 5.1 yields \( \text{rep}(Q_n) = \varphi_n/3 \). On the other hand, if there exists \( n_0 \) such that \( r = \text{rep}(Q_{n_0}) \) is divisible by 3, then \( r \) cannot be the product of more than \( n - 1 \) distinct primes, since this would imply \( r \geq \varphi_n \), a contradiction to Theorem 5.1. Thus, \( r = 2 \cdot 3 \cdot q_3 \cdot \ldots \cdot q_{n-1} \), where \( q_3, \ldots, q_{n-1} \) are primes and \( 5 \leq q_3 < \ldots < q_{n-1} \). Let \( k \) be the number of primes in \([5, q_{n-1}]\) distinct from \( q_3, \ldots, q_{n-1} \). By iterated application of Lemma 5.2, we see that for \( \ell \geq k \), \( Q_{n+\ell} \) is representable modulo \( \varphi_{n+\ell-1} \). Since \( \text{rep}(Q_{n+\ell}) \geq \varphi_{n+\ell-1} \) by Proposition 5.3, it follows that \( \text{rep}(Q_{n+\ell}) = \varphi_{n+\ell-1} \) for all \( \ell \geq k \).
References


