Pontryagin products and adequate equivalence relations

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Abstract

Let \( A \) be an abelian variety over an algebraically closed field. The Chow group \( CH_0(A) \) of zero-dimensional cycles modulo rational equivalence forms a ring under the Pontryagin product operation with respect to which the subset \( I \) of degree zero cycles is an ideal. The \( n \)th power of \( I \) is shown to agree with the \( n \)th power of the algebraic equivalence relation as defined by H. Saito. A result is also proven relating products of equivalence relations (in the relative setting) to the decomposition of the Chow motive of an abelian variety.

1 Introduction

The notion of an adequate equivalence relation on the group of cycles on an algebraic scheme was first defined by Samuel in his 1960 paper [Sam]. More recently, Hiroshi Saito defined a product structure on the set of adequate equivalence relations; this operation was studied in detail in [Sai] and [Ja]. Let \( A \) be an abelian variety over an algebraically closed field and \( I \subseteq CH_0(A) \) the group of zero-dimensional cycles on \( A \) of degree 0 modulo rational equivalence. Using the group law on the abelian variety, one may define a “Pontryagin product”, endowing \( CH_0(A) \) with a ring structure for which \( I \) is an ideal.

This work is divided into two parts. In the first section, our main result is that the \( n \)th ideal power of \( I \) coincides with the group of zero-cycles on \( A \) which are equivalent to zero modulo the equivalence relation defined by the \( n \)th power of algebraic equivalence. The proof uses Beauville’s eigenspace decomposition of the Chow groups of \( A \) (cf. [Be2]) in conjunction with some results describing the structure of \( I \); we work with \( \mathbb{Z} \)-coefficients throughout this argument. In the second section, we study the relationship between (rational) cycles on \( CH^g(A \times_k A) \) (where \( g = \dim A \)) and a relative version of the notion of adequate equivalence relation; this is intimately connected to the Künneth decomposition of the Chow motive \( h(A) \) [DM], [Kü].
results here are similar in spirit to those of [Ja] concerning the relationship between adequate equivalence relations and the (conjectural) Bloch-Beilinson filtration.

Throughout this paper an algebraic scheme is a scheme of finite type over a field $k$, while a variety is an integral algebraic scheme. A variety is said to be defined over $k$ if it is geometrically integral.

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2 Preliminaries

2.1 Cycles and the Pontryagin Product

Let $k$ be a field and $X$ an algebraic scheme over $k$. We denote by $Z_i(X)$ the group of $i$-dimensional cycles on $X$, that is, the free abelian group on the set of dimension $i$ subvarieties of $X$. We denote by $CH_i(X)$ the Chow group of $i$-dimensional cycles; that is, $Z_i(X)$ modulo the subgroup of cycles rationally equivalent to zero, and set $Z_*(X) = \bigoplus_i Z_i(X)$, $CH_*(X) = \bigoplus_i CH_i(X)$. The class of a subvariety $V \subseteq X$ in $CH^*(X)$ is denoted $[V]$. If $X$ is equidimensional, we denote by $Z_j(X)$ (resp. $CH_j(X)$) the Chow group of codimension $j$ cycles (resp. codimension $j$ cycles modulo rational equivalence) on $X$; clearly, $Z_j(X) = Z_{d-j}(X)$ and $CH_j(X) = CH_{d-j}(X)$ where $d = \dim X$. It is well-known [Fu] that the graded group $CH^*(X) = \bigoplus_i CH_i(X)$ may be endowed with the structure of commutative graded ring under intersection product. Following convention, we will denote the intersection of two cycles $\alpha, \beta \in CH^*(X)$ by $\alpha \cdot \beta$. If $\alpha \in CH^*(X)$ and $\gamma \in CH^*(Y)$ are cycles, we denote by $x \times y \in CH^*(X \times_k Y)$ the cycle $p_1^*(\alpha) \cdot p_2^*(\gamma)$, where $p_1: X \times_k Y \to X$ and $p_2: X \times_k Y \to Y$ are the projection maps. If $X, Y$, and $Z$ are smooth and projective, and $\alpha \in CH^*(X \times_k Y)$, $\beta \in CH^*(Y \times_k Z)$, we denote by $\beta \circ \alpha$ the convolution product $(p_{13})_*(p_{12}^* \alpha \cdot p_{23}^* \beta) \in CH^*(X \times_k Z)$. (Here $p_{ij}$ is the projection of $X \times_k Y \times_k Z$ on the $(i, j)$th factor. If $R$ is any ring, we write $CH^*(X; R)$ as shorthand for $CH^*(X) \otimes R$, etc.

Now suppose $A$ is an abelian variety and $\mu: A \times_k A \to A$ is the morphism giving the group law on $A$. One may then define a product structure, namely the Pontryagin product as follows:

$$*: CH_r(A) \otimes CH_s(A) \to CH_{r+s}(A)$$

$$\alpha \times \beta \mapsto \alpha * \beta := \mu_*(\alpha \times \beta)$$
Clearly $CH_0(A)$ is a subring of $CH_\ast(A)$ for this ring structure. In the sequel, we will often use formal sums (as in cycle groups) and addition of points on the abelian variety in the same formula; in an attempt to dispel potential confusion arising from this, we will denote the former by the ordinary summation symbol $\sum$ and the latter by $\sum^A$.

Now let $k$ be an algebraically closed field. Consider the degree map $\deg : CH_0(A) \to \mathbb{Z}$, and let $I = \text{Ker} \deg$. Then $I$ is generated by cycles of the form $[a] - [0]$ (where $a \in A$ is a closed point and $0 \in A$ is the identity) and is an ideal of $CH_0(A)$ with respect to Pontryagin product. We denote by $I^{\ast n}$ the Pontryagin powers of the ideal $I$.

An elementary argument gives the following, cf. [Bl2].

**Lemma 2.1.** Let $\text{alb} : I \to A$ be the Albanese map of $A$, i.e. $\text{alb}(\sum [P_i]) = \sum^A P_i$. Then there is an exact sequence

$$0 \longrightarrow I^{\ast 2} \longrightarrow I \overset{\text{alb}}{\longrightarrow} A \longrightarrow 0$$

Other important properties of the ideal $I$ are summarized below:

**Proposition 2.2.**

1. (Bloch, [Bl1] Lemma 1.4) \newline
   $I$ is divisible.

2. (Roitman, [Ro]) \newline
   $I^{\ast 2}$ is uniquely divisible.

3. (Bloch, [Bl2], Theorem 0.1) \newline
   $I^{\ast (g+1)} = 0$.

Since $I^{\ast n}$ is generated by products from $I$, it follows immediately from the above that $I^{\ast n}$ is uniquely divisible for all $n \geq 2$.

The following lemma will be necessary in the proof of our main result. The first assertion is elementary and follows from the definitions; the second is standard and may be proved by induction.
Lemma 2.3. Let $A$ be an abelian variety and $a \in A$ a closed point. Let $\tau_a : A \to A$ denote the translation map $x \mapsto x + a$

1. For any $z \in \text{CH}_*(A)$, $(\tau_a)_* z = [a] * z$, $(\tau_a)^* z = [-a] * z$.

2. For any $a_1, \ldots, a_n \in A$,

\[
([a_1] - [0]) * \cdots * ([a_n] - [0]) = \sum_{e_1, \ldots, e_n \in \{0, 1\}} (-1)^{e_1 + \cdots + e_n} \sum_{i=0}^A a_i
\]

One of the most significant results in the study of the Chow groups of an abelian variety is the following theorem on the diagonalizability of the multiplication by $n$ morphism on $A$. We state it here in a general form, valid for abelian schemes over any smooth quasiprojective base, as we will adopt this perspective in a later section.

Theorem 2.4. (Beauville, [Be2]; Deninger-Murre [DM], Theorem 2.19) Let $k$ be a field, $S$ a smooth quasiprojective algebraic $k$-scheme, and $A$ an abelian scheme over $S$. Let $d$ be the dimension of $S$ and $g$ the fiber dimension of $A$ over $S$. Let $n : A \to A$ denote the $(S)$-morphism representing multiplication by $n$ on $A$. For each $s \in \mathbb{Z}$ define:

\[
\text{CH}^i_s(A; \mathbb{Q}) := \{ x \in \text{CH}^i(A; \mathbb{Q}) : n^s x = n^{2i-s} x \}
\]

Then there is an isomorphism

\[
\text{CH}^i(A; \mathbb{Q}) \cong \bigoplus_{s=p'}^{p''} \text{CH}^i_s(A; \mathbb{Q})
\]

where $p' = \min(2p, p + d)$ and $p'' = \max(p - g, 2(p - g))$.

Furthermore, $\text{CH}^i_s(A; \mathbb{Q}) = 0$ if $s < 2p - 2g$ or $s > 2p$.

The proof of this theorem involves Fourier theory for Chow groups. We will also need the following functorial properties of the eigenspaces $\text{CH}^i_s(A; \mathbb{Q})$; proofs may be found in [Be2].

Proposition 2.5. (Beauville [Be2], Deninger-Murre [DM])

1. Let $A$ be an abelian variety over a field and $J$ the image of $I$ under the natural map $q : \text{CH}^g(A) \to \text{CH}^g(A; \mathbb{Q})$. Then $J^{sr} = \bigoplus_{s=r}^g \text{CH}^g_s(A; \mathbb{Q})$. 

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2. Let $A$ be an abelian scheme of fiber dimension $g$ over $S$. If $x \in CH^p_s(A; \mathbb{Q})$ and $y \in CH^q_s(A; \mathbb{Q})$, then $x \cdot y \in CH^{p+q}_{s+t}(A; \mathbb{Q})$ and $x \ast y \in CH^{p+q-g}_{s+t}(A; \mathbb{Q})$.

3. Let $f: A \to B$ be a homomorphism of abelian schemes over $S$. Then

$$f^*(CH^p_s(B; \mathbb{Q})) \subseteq CH^p_s(A; \mathbb{Q})$$

and

$$f_*(CH^p_s(A; \mathbb{Q})) \subseteq CH^{p+c}_s(B; \mathbb{Q}),$$

where $c = \dim B - \dim A$.

2.2 Adequate Equivalence Relations

Let $k$ be a field and $V_k$ the class of smooth projective varieties over $k$. The following definition is due (at least in the case $C = V_k$) to Samuel:

**Definition 2.6.** Let $C$ be a full subcategory of $V_k$. An adequate equivalence relation on $C$ is an assignment, to every object $X$ of $C$, of a graded subgroup $EZ^*(X) \subseteq Z^*(X)$ with the following properties:

1. If $\alpha, \beta \in Z^*(X)$, then there exists a cycle $\alpha' \in Z^*(X)$ such that $\alpha'$ and $\beta$ intersect properly, and $\alpha - \alpha' \in EZ^*(X)$.

2. Let $\alpha \in Z^*(X)$ and $\beta \in Z^*(X \times_k Y)$ ($X, Y$ objects of $C$) such that the intersection $\beta \cap (\alpha \times_k Y)$ is defined. If $\alpha \in EZ^*(X)$, then $(p_2)_*(\beta \cap p_1^*(\alpha)) \in EZ^*(Y)$, where $p_1$ and $p_2$ are the respective projections of $X \times Y$ onto the first and second factors.

Essentially, the first condition implies that some sort of moving lemma holds for $E$-equivalence, and the second condition guarantees preservation of $E$-equivalence under convolution, or, equivalently, pull-backs, pushforwards, and intersection products. If we do not specify the subcategory $C$, we will assume without further comment that $C = V_k$.

Rational equivalence is perhaps the most common adequate equivalence relation. Historically, algebraic equivalence, homological equivalence (with respect to some Weil cohomology theory, cf. [Kl]) and numerical equivalence have also been studied. If $E$ and $E'$ are two equivalence relations, we say that $E$ is finer than $E'$ if $EZ^*(X) \subseteq E'Z^*(X)$ for all $X \in V_k$. The following theorem summarizes some well-known relationships among the equivalence relations mentioned above.

**Theorem 2.7.**

- Rational equivalence is strictly finer than algebraic equivalence, which is strictly finer than homological equivalence, which in turn is strictly finer than numerical
equivalence. With \( \mathbb{Q} \)-coefficients, Grothendieck’s standard conjectures predict that numerical equivalence and homological equivalence (with respect to any Weil cohomology theory) coincide [Kl].

- (Samuel, [Sam]) Rational equivalence is the finest adequate equivalence relation.
- With \( \mathbb{Q} \)-coefficients, numerical equivalence is the coarsest non-trivial adequate equivalence relation.

Another important (adequate) equivalence relation, called \( \ell \)-cubical equivalence, was defined by Samuel [Sam]:

**Definition 2.8.** Let \( k \) be an algebraically closed field and \( \ell \geq 0 \) an integer. Two cycles \( \alpha_1, \alpha_2 \in \mathbb{Z}^i(X) \) are called \( \ell \)-cubically equivalent if there exist curves \( C_1, \ldots, C_\ell \), a cycle \( z \in \mathbb{Z}^i(C_1 \times_k \cdots \times_k C_\ell \times_k X) \) and closed points \( p_1^i, p_\ell^i \in C_i \) (\( i = 1, \ldots, \ell \)) such that \( s(p_1^i, \ldots, p_\ell^i)^*(Z) \in \mathbb{Z}^i(X) \) (pullback of \( Z \) via the inclusion \( s(p_1^i, \ldots, p_\ell^i) : \text{Spec} \, k \hookrightarrow C_1 \times_k \cdots \times_k C_\ell \) of the closed point \( p = (p_1^e, \ldots, p_\ell^e) \in C_1 \times_k \cdots \times_k C_\ell \)) exists for all \( e_1, \ldots, e_\ell \in \{0, 1\} \) and such that

\[
\alpha_1 - \alpha_2 = \sum_{e_1, \ldots, e_\ell \in \{0, 1\}} (-1)^{e_1 + \cdots + e_\ell} s(p_1^{e_1}, \ldots, p_\ell^{e_\ell})^*(Z).
\]

As noted in [Ja], p. 229, a Bertini-type argument implies that the same equivalence relation is obtained if one replaces the “parameter varieties” \( C_i \) above by arbitrary smooth projective varieties, or by abelian varieties; one may even take \( C_1, \ldots, C_\ell \) to be the same curve.

Let \( F_\ell \mathbb{Z}^*(X) \) denote the group of cycles \( \ell \)-cubically equivalent to zero. It is clear from the definition that \( F_\ell \mathbb{Z}^*(X) \) coincides with the subgroup of cycles algebraically equivalent to zero, which we henceforth denote \( \mathbb{L} \mathbb{Z}^*(X) \).

In light of the fact that rational equivalence is the finest adequate equivalence relation, it is often convenient to adopt the following notation: given an adequate equivalence relation \( E \), let \( EC^*(X) \) denote the image of \( EZ^*(X) \) under the quotient map \( Z^*(X) \twoheadrightarrow CH^*(X) \). Then giving an adequate equivalence relation \( E \) is equivalent to specifying subgroups \( EC^*(X) \) which are respected by pushforwards and pullbacks via appropriate morphisms \( f : X \to Y \) and such that \( \alpha \in CH^*(X), \beta \in EC^*(X) \implies \alpha \cdot \beta \in CH^*(X) \). (cf. [Ja], Lemma 1.3) Equivalently, we could stipulate simply that the subgroups \( EC^*(X) \) are preserved under convolution of cycles.

Hiroshi Saito [Sai] has defined the following notion of product of equivalence relations. For convenience, we work, as before, modulo rational equivalence.
Definition 2.9. Let $E$ and $E'$ be adequate equivalence relations. A cycle $\alpha \in CH^*(X)$ is said to be $(E \ast E')$-equivalent to zero (we write $\alpha \in (E \ast E')CH^*(X)$) if $\alpha$ is a sum of cycles of the form $p_*(\alpha_1 \cdot \alpha_2)$, where $T$ is a smooth projective variety, $\alpha_1 \in ECH^*(X \times_k T)$, $\alpha_2 \in E' * (X \times_k T)$ and $p : X \times_k T \rightarrow X$ is projection on the first factor.

Proposition 2.10. $E \ast E'$ is an adequate equivalence relation, which is finer than both $E$ and $E'$.

This product operation is evidently associative (and commutative); hence we may speak of the $n$th power $E^\ast n$ of $E$ for any $n \geq 1$; by convention $E^\ast 0$ is the trivial relation. An important observation proceeding straight from the definition and linking two of the examples above is:

Proposition 2.11. The $\ell$-cubical equivalence relation is the $\ell$th power of the algebraic equivalence relation, i.e. $F_\ell = L^\ast \ell$

3 Zero-cycles on an abelian variety

Let $A$ be an abelian variety defined over an algebraically closed field $k$. It is well-known that $I = \text{Ker} (\deg : CH_0(A) \rightarrow \mathbb{Z})$ coincides with the subgroup of zero-dimensional cycles algebraically equivalent to zero.

Our main result is:

Theorem 3.1. For any $n \geq 0$,

$$I^\ast n = L^\ast n CH_0(A)$$

In particular, $L^\ast 2 CH_0(A) = I^\ast 2 = \text{Ker} (\text{alb} : I \rightarrow A)$, and if $n > g = \text{dim} A$, then $L^\ast n CH_0(A) = 0$.

For emphasis, we note that the $\ast$ on the left represents the Pontryagin power of the ideal $I$, while the $\ast$ on the right represents the power of $L(\text{algebraic equivalence})$ as an (adequate) equivalence relation. Note also that, in contrast to [Ja], we work with integral, not rational coefficients.

Proof. When $n = 0$, the statement is trivial, and when $n = 1$, the assertion is that $I$ is equal to the group of cycles algebraically equivalent to zero; this is well-known ([Fu], Chapter 17).
Thus $\tau$ shows that our original cycle $\ell$ is a member of $F_n \cdot CH_0(A)$. Hence the class of the cycle

$$\sum_{e_1, \ldots, e_n \in \{0,1\}} (-1)^{e_1 + \ldots + e_n} s(p_1^{e_1}, \ldots, p_n^{e_n})^*(Z) = \sum_{e_1, \ldots, e_n \in \{0,1\}} (-1)^{e_1 + \ldots + e_n} \sum_{i \in \mathbb{N}} a_i$$

modulo rational equivalence is in fact equal to $([a_1] - [0]) \ldots ([a_n] - [0])$. This shows that our original cycle $c$ is a member of $F_n \cdot CH_0(A)$.

Conversely, suppose $c$ is a generator of $F_n \cdot CH_0(A)$. By the remark following Definition 2.8, we may assume that the “parameter varieties” are all abelian varieties. Thus, there are abelian varieties $A_1, \ldots, A_n$, a subvariety $Z \subseteq A \times_k A_1 \times_k \ldots \times_k A_n$ and points $p^0_i, p^1_i \in A_i$ for $i = 1, \ldots, n$ such that

$$c = \sum_{e_1, \ldots, e_n \in \{0,1\}} (-1)^{e_1 + \ldots + e_n} s(p_1^{e_1}, \ldots, p_n^{e_n})^*(Z)$$

Without loss of generality, we may assume that $p^1_i = 0 \in A_i$ for all $i = 1, \ldots, n$.

Recall that $s(p_1^{e_1}, \ldots, p_n^{e_n})$ represents pullback via the point $(p_1^{e_1}, \ldots, p_n^{e_n})$. Letting $\tau(p_1^{e_1}, \ldots, p_n^{e_n}) : A_1 \times_k \ldots \times_k A_n \to A_1 \times_k \ldots \times_k A_n$ denote the translation map $z \mapsto z + (p_1^{e_1}, \ldots, p_n^{e_n})$, we have

$$s(p_1^{e_1}, \ldots, p_n^{e_n}) = (id_A \times_k \tau(p_1^{e_1}, \ldots, p_n^{e_n})) \circ s(0, \ldots, 0)$$

Thus

$$c = \sum_{e_1, \ldots, e_n \in \{0,1\}} (-1)^{e_1 + \ldots + e_n} s(p_1^{e_1}, \ldots, p_n^{e_n})^*([Z])$$

$$= s(0, \ldots, 0)^* \sum_{e_1, \ldots, e_n \in \{0,1\}} (-1)^{e_1 + \ldots + e_n} \tau(p_1^{e_1}, \ldots, p_n^{e_n})^*([Z])$$

$$= s(0, \ldots, 0)^* \sum_{e_1, \ldots, e_n \in \{0,1\}} (-1)^{e_1 + \ldots + e_n} \tau(-p_1^{e_1}, \ldots, -p_n^{e_n})^*([Z])$$
\[
\begin{aligned}
&= s(0, \ldots, 0)^* \{[[0, 0, \ldots, 0]] - [[0, 0, \ldots, 0]]\} \ast \ldots \\
&\ldots \ast \{[[0, 0, \ldots, 0]] - [[0, 0, \ldots, 0, p_n]]\} \ast \{[Z]\}
\end{aligned}
\]

Now consider the expression above as an element of \(CH^p(A, \mathbb{Q})\). Since each zero-cycle in curly braces above has degree 0, its image under the map \(q : I \rightarrow J\) (introduced in Proposition 2.5) is a member of \(\oplus_{s \geq 1} CH^p_s(A, \mathbb{Q})\) by Proposition 2.5, part 1. Thus, by part 2 of Proposition 2.5, the image under \(q\) of the expression in the outer parentheses lies in \(\oplus_{s \geq n} CH^p_s(A \times_k A_1 \times_k \ldots \times_k A_n, \mathbb{Q})\). Finally, by part 3 of Proposition 2.5, applying \(s(0, \ldots, 0)^*\) we conclude that \(q(c)\) is an element of \(\oplus_{s \geq n} CH^p_s(A), \mathbb{Q}\), which may be identified with \(J^{*n}\) by part 1 of Proposition 2.5. For every \(n \geq 1\), \(q : I \rightarrow J\) restricts to a map \(q_n : I^{*n} \rightarrow J^{*n}\). However, by Roitman’s Theorem (Theorem 2.1, part 2) \(I^{*n}\) is uniquely divisible for \(n \geq 2\), so \(q_n\) is an isomorphism; hence, \(c \in I^{*n}\) as desired.

**Corollary 3.2.** Let \(C\) be a smooth projective curve over an algebraically closed field. Then \(LCH_0(C) = \text{Ker}(\deg : CH_0(C) \rightarrow \mathbb{Z})\) and \(L^{*n}CH_0(C) = 0\) for \(n \geq 2\).

**Proof.**
The first assertion is classical. For the second, let \(J\) be the Jacobian of \(C\) and \(\iota : C \hookrightarrow J\) the associated map. Then functoriality of the Albanese map yields a commutative diagram:

\[
\begin{array}{ccc}
LCH_0(C) & \xrightarrow{\text{alb}_C} & \text{Alb}(C)(k) = J(k) \\
\downarrow{\iota_*} & & \downarrow{=} \\
LCH_0(J) & \xrightarrow{\text{alb}_J} & \text{Alb}(J)(k) = J(k)
\end{array}
\]

Since \(L^{*n}\) is adequate, \(\iota_*(L^{*n}CH_0(C)) \subseteq L^{*n}CH_0(J) \subseteq L^{*2}CH_0(J) = (\text{Ker alb}_J)\). By commutativity of the diagram, \(\text{alb}_C(L^{*n}CH_0(C)) = 0\), but \(\text{alb}_C\) is an isomorphism, whence the result.

If we allow ourselves \(\mathbb{Q}\)-coefficients, the method employed in the second half of the proof of Theorem 3.1 may be modified to prove the following statement on the relationship between algebraic equivalence and the eigenspace decomposition, first conjectured by Beauville ([Be1], p. 258):

**Proposition 3.3.** Let \(A\) be an abelian variety of dimension \(g\) and \(0 \leq p \leq g\) an integer. Then \(LCH^p(A) \otimes \mathbb{Q} \subseteq \bigoplus_{s=p-g+1}^p CH^p_s(A; \mathbb{Q})\)

**Proof.**
Suppose $x \in LCH^p(A)$. Without loss of generality, we may assume that there is an abelian variety $B$ (of dimension $g'$, a point $b \in B$, and a cycle $Z \subseteq A \times_k B$, such that $x = s_b^*([Z]) - s_0^*([Z])$, where for any point $p \in B$, $s_p : A \to A \times_k B$ is the map $a \mapsto (a, p)$. As before, we have: $x = s_b^*([Z]) - s_0^*([Z]) = s_0^*(([0, 0]) - [(0, b)]) * [Z]$. Since $[(0, 0)] - [(0, b)]$ is a zero-cycle of degree 0 on $A \times_k B$, by part 1 of Proposition 2.5 it is an element of $\bigoplus_{s \geq 1} CH^{g+g'}(A \times_k B; \mathbb{Q})$. In view of the isomorphism (Theorem 2.4) $CH^p(A; \mathbb{Q}) \cong \bigoplus_{s=g-p} CH^p_s(A; \mathbb{Q})$, it follows from part 2 of Proposition 2.5 that $x \in \bigoplus_{s=g-p+1} CH^p(A; \mathbb{Q})$.

4 Küneth decomposition and adequate equivalence relations

We recall the following theorem giving a Küneth decomposition of the class of the diagonal of an abelian variety. In the interest of keeping the exposition self-contained, we will refrain from explicit mention of Chow motives and instead refer the reader to [DM] and [Kü] for details.

**Theorem 4.1.** (Deninger-Murre, Theorem 3.1; Künemann, Theorem 3.1.1)

Let $S$ be a smooth quasiprojective scheme over a base field $k$ and $B/S$ an abelian scheme of fiber dimension $n$. Let $\Delta$ be the diagonal of $B$; that is, the graph of the identity morphism $B \to B$. There is a unique decomposition:

$$[\Delta] = \sum_{i=0}^{2n} \pi_i \text{ in } CH^n(B \times_S B)$$

such that $(id_B \times n)^* \pi_i = n^i \pi_i$ for each $i$ and all $n \in \mathbb{Z}$. Furthermore, $\pi_i \circ \pi_j = 0$ for $i \neq j$, $\pi_i \circ \pi_i = \pi_i$ for all $i$, and for each $i$, $\sigma^*(\pi_i) = \pi_{2g-i}$, where $\sigma : B \times_S B \to B \times_S B$ is the transposition of factors.

In fact, Künemann has given the following explicit formula for $\pi_i$ ([Kü], p. 200):

$$\pi_i = \frac{1}{(2g-i)!} (\log([\Delta])^{(2g-i)})$$

where

$$\log([\Delta]) = \sum_{n=1}^{\infty} (-1)^n \frac{([\Delta] - [\Gamma_e])^n}{n},$$

$\Gamma_e$ is the graph of the map $B \to B$ sending everything to the identity section of $B$, and $*$ represents Pontryagin product on $B \times_S B$, considered as an abelian $B$-scheme.
via projection on the first factor. Only finitely many of the terms in the series defining log([Δ]) are nonzero (cf. [Kü], Theorem 1.4.1), so this expression is well-defined. It follows readily from the definitions that \( CH^n(B \times_S B) \) is a (noncommutative) ring under convolution of cycles; the above theorem asserts that the unit element for this ring structure may be decomposed as a sum of mutually orthogonal idempotents (“projectors”), each of which is an eigenvector for the maps \( id_B \times_\mathbf{n} \).

Now let \( k \) be any field and \( A \) a (fixed) abelian variety over \( k \); set \( g = \dim A \). Let \( \mathcal{V}_k/A \) denote the full subcategory of \( \mathcal{V}_k \) determined by objects of the form \( A \times_k T \) where \( T \) is any smooth projective variety over \( k \). Viewing \( A \times_k T \) as an abelian \( T \)-scheme, Theorem 2.4 gives a decomposition (in which some of the eigenspaces are zero):

\[
CH^p(A \times_k T; \mathbb{Q}) \cong \bigoplus_{s=2p-2g}^{2p} CH^p_s(A \times_k T; \mathbb{Q})
\]

The following statement relates the eigenspaces to convolution with the projectors defined above.

**Proposition 4.2.** For any \( i, 0 \leq i \leq 2g \) and any \( p \),

\[
CH^p(A \times_k T; \mathbb{Q}) \circ \pi_i = CH^p_{2p-2g+i}(A \times_k T; \mathbb{Q})
\]

**Proof.**

Let \( p_{ij} \) denote the projection from map from \( A \times_k A \times_k T \) onto the \((i,j)\)th factor. Then for \( \alpha \in CH^p(A \times_k T; \mathbb{Q}) \),

\[
(n \times 1)^*(\alpha \circ \pi_i) = (n \times 1)^*((p_{13})_*(p_{12}^*\pi_i \cdot p_{23}^*\alpha))
\]

\[
= (p_{13})_*((n \times 1 \times 1)^*(p_{12}^*\pi_i \cdot p_{23}^*\alpha))
\]

\[
= (p_{13})_*(p_{12}^*(n \times 1)^*\pi_i \cdot p_{23}^*\alpha)
\]

\[
= n^{2g}(\alpha \circ \pi_i)
\]

in view of the equalities:

\[
(n \times 1)^*\pi_i = (\sigma \times 1)^*(1 \times n)^*(\sigma \times 1)^*\pi_i = (\sigma \times 1)^*(1 \times n)^*\pi_{2g-i} = (\sigma \times 1)^*(n^{2g-i}\pi_{2g-i})
\]

\[
= n^{2g-i}\pi_i
\]

Thus, \( CH^p(A \times_k T; \mathbb{Q}) \circ \pi_i \subseteq CH^p_{2p-2g+i}(A \times_k T; \mathbb{Q}) \).
For the other inclusion, observe that

\[ CH_{2p-2g+i}(A \times_k T; Q) = CH_{2p-2g+i}(A \times_k T; Q) \circ [\Delta] = CH_{2p-2g+i}(A \times_k T; Q) \circ \sum_{j=0}^{2g} \pi_j \]

From the inclusion just proved, \( CH^p(A \times_k T; Q) \circ \pi_j \subseteq CH_{2p-2g+i}(A \times_k T; Q) \); hence the expression on the previous line collapses to:

\[ CH_{2p-2g+i}(A \times_k T; Q) \circ \pi_i \]

as desired.

Our main result is:

**Theorem 4.3.** For each integer \( r \geq 0 \), set

\[ F_r CH^p(A; Q) = \bigoplus_{s=2p-2g}^{2p-r} CH^p_s(A; Q) \]

1. \( F^r \) defines an adequate equivalence relation on \( V_k/A \).

Furthermore, \( F^a \cdot F^b \subseteq F^{a+b} \).

2. \( F^r = (F^1)^r \) as adequate equivalence relations on \( V_k/A \).

**Proof.**

We prove first that \( F^r \) is preserved under pullbacks and pushforwards; then we show \( F^a \cdot F^b \subseteq F^{a+b} \). This will suffice to show that \( F^r \) is adequate.

Let \( f : A \times_k T \longrightarrow A \times_k S \) be a morphism in \( V_k/A \). Set \( d_T = \dim T \), \( d_S = \dim S \).

Then for \( \alpha \in CH^p(A \times_k S; Q) \), we have

\[ (n \times 1_S)^* f^*(\alpha) = f^*(n \times 1_T)^*(\alpha) = n^{2p-s} f^*(\alpha) \]

Thus,

\[ f^*(F^r CH^p(A \times_k S; Q)) = f^*(\bigoplus_{s=2p-2g}^{2p-r} CH^p_s(A \times_k S; Q)) \subseteq \bigoplus_{s=2p-2g}^{2p-r} CH^p_s(A \times_k T; Q) = F^r CH^p(A \times_k S; Q) \]

Furthermore, for \( \beta \in CH^p_r(A \times_k T; Q) \), we have

\[ (n \times 1_S)^* f_*(\beta) = f_*(n \times 1_T)^* \beta = n^{2p-t} f_*(\beta) \in CH^{p+t+2(d_S-d_T)}_{i+2(d_S-d_T)}(A \times_k S; Q) \]
Therefore
\[ f_*(F^rCH^p(A \times_k T; \mathbb{Q})) = f_* \left( \bigoplus_{t=2p-2g}^{2p-r} CH^p_t(A \times_k T; \mathbb{Q}) \right) \]
\[ \subseteq \bigoplus_{t=2(p+g-dT)-2}^{2(p+g-dT)-r} CH^p_t(A \times_k S; \mathbb{Q}) = F^rCH^{p+dS-dT}(A \times_k S; \mathbb{Q}) \]

Finally, if \( \alpha \in CH^p(A \times_k S) \) and \( \beta \in CH^q(A \times_k S) \), then

\[ (n \times 1)^*(\alpha \cdot \beta) = (n \times 1)^*(\alpha) \cdot (n \times 1)^*\beta = n^{2(p+q)-(s+1)}(\alpha \cdot \beta) \]

Thus,

\[ F^aCH^p(A \times_k S; \mathbb{Q}) \cdot F^bCH^q(A \times_k S; \mathbb{Q}) = \bigoplus_{s=2p-2g}^{2p-a} CH^p_s(A \times_k S; \mathbb{Q}) \cdot \bigoplus_{t=2q-2g}^{2q-b} CH^q_t(A \times_k S; \mathbb{Q}) \]
\[ \subseteq \bigoplus_{u=2(p+q)-4g}^{2(p+q)-(a+b)} CH^{p+q}_u(A \times_k S; \mathbb{Q}) = \bigoplus_{u=2(p+q)-2g}^{2(p+q)-(a+b)} CH^{p+q}_u(A \times_k S; \mathbb{Q}) \]
\[ = F^{a+b}CH^{p+q}(A \times_k S; \mathbb{Q}) \]

since the eigenspaces \( CH^{p+q}_u(A \times_k S; \mathbb{Q}) \) are zero for \( s < 2(p+q) - 2g \).

For the second assertion, we need the following:

**Lemma 4.4.** Consider \( A \times_k A \) as an \( A \)-scheme via projection on the first factor. Then \( \pi_i \in (F^{1})^{*(2g-1)}CH^g(A \times_k A; \mathbb{Q}) \).

By Künnemann’s formula (following Theorem 4.1), we have

\[ \pi_{2g-r} = \frac{1}{r!} \pi_{2g-1}^{2g-r} \]

in which \( * \) represents Pontryagin product on \( A \times_k A \), considered as an \( A \)-scheme via projection on the first factor. For convenience, we shall use the notation \( A \) to denote \( A \times_k A \), considered as an abelian \( A \)-scheme in the above description.

Since \( (n \times 1)^*\pi_i = n^i \pi_i \) by Theorem 4.1, we have \( \pi_i \in CH^g_{2g-1}(A; \mathbb{Q}) \); in particular, \( \pi_{2g-1} \in F^1(A; \mathbb{Q}) \).

Next, let \( \hat{A} \) denote the abelian \( A \)-scheme dual to \( A \). There are “Fourier transform” homomorphisms (see [DM] or [Küi]):

\[ \mathcal{F} : CH^*(A; \mathbb{Q}) \rightarrow CH^*(\hat{A}; \mathbb{Q}), \quad \hat{\mathcal{F}} : CH^*(\hat{A}; \mathbb{Q}) \rightarrow CH^*(A; \mathbb{Q}) \]
given by convolution, i.e. there are elements $F \in CH^*(A \times A)$ and $\hat{F} \in CH^*(\hat{A} \times A)$ such that $\mathcal{F}(x) = F \circ x$ and $\mathcal{F}(y) = \hat{F} \circ y$. Furthermore, for any $x \in CH^*(A; \mathbb{Q})$ and $y \in CH^*(\hat{A}; \mathbb{Q})$, there are formulas:

$$
\mathcal{F}(\alpha \ast \beta) = \mathcal{F}(\alpha) \cdot \mathcal{F}(\beta) \quad \text{and} \quad \mathcal{F}(\alpha) = (-1)^{g-1} \mathcal{F}(\alpha^*)
$$

Since $\pi_{g-1} = F^1 CH^*(A; \mathbb{Q})$ and $F^1$ is adequate, it follows that $\mathcal{F}(\pi_{g-1}) = F \circ \pi_{g-1} \in F^1 CH^*(A; \mathbb{Q})$. Thus for any $i \geq 1$,

$$
\mathcal{F}(\pi_{g-i}) = \mathcal{F}(\frac{1}{i!} \pi_{2g-i}) = \frac{1}{i!} (\mathcal{F}(\pi_{g-1}))^i \in (F^1)^i CH^*(A; \mathbb{Q})
$$

by the definition of the product of two (adequate) equivalence relations. Finally, because $F^*i$ is adequate by Proposition 2.10, we have

$$
\pi_{g-i} = (-1)^{g-i} \mathcal{F}(\pi_{g-i}) \in (F^1)^i CH^*(A; \mathbb{Q})
$$

which completes the proof of the Lemma.

Returning to the proof of Theorem 4.3, the inclusion $(F^1)^{sr} \subseteq F^r$ may be proved by induction on $r$, the case $r = 1$ being trivial. Evidently, $(F^1)^{(r+1)} = (F^1)^{sr} \ast F^1$, which by the induction hypothesis equals $F^r \ast F^1$. Now if $\gamma \in (F^r \ast F^1) CH^*(A \times_k S)$, there exists a smooth projective variety $T$ and elements $\alpha \in F^r CH^*(A \times_k S \times_k T)$, $\beta \in F^1 CH^*(A \times_k S \times_k T)$ such that $\gamma = p_*(\alpha \cdot \beta)$ where $p : A \times_k S \times_k T \rightarrow A \times_k S$ is the projection map. From definitions, it is clear that $\alpha \cdot \beta \in F^{r+1} CH^*(A \times_k S \times_k T)$, and since $F^{r+1}$ is adequate (or by Proposition 2.5, part 3), it follows that $\gamma = p_*(\alpha \cdot \beta) \in F^{r+1} CH^*(A \times_k S)$.

For the reverse inclusion, suppose

$$
\alpha \in F^r CH^p(A \times_k A; \mathbb{Q}) \quad \text{and} \quad \beta \in F^{2p-r} CH^p(A \times_k A; \mathbb{Q}) \quad \text{where} \quad \alpha = \bigoplus_{s=2p-2g}^{2p-r} CH^p(A \times_k A; \mathbb{Q}) \circ \pi_{s+2g-2p}
$$

Since $\pi_{s+2g-2p} \in (F^1)^{(2p-s)} CH^g(A \times_k A; \mathbb{Q})$, it follows from adequacy of $(F^1)^{(2p-s)}$ that $CH^p(A \times_k A; \mathbb{Q}) \circ \pi_{s+2g-2p} \in F^{2p-s} CH^p(A \times_k A; \mathbb{Q})$. Thus

$$
\alpha \in \bigoplus_{s=2p-2g}^{2p-r} (F^1)^{(2p-s)} CH^p(A \times_k A; \mathbb{Q}) = \bigoplus_{t=r}^{2g} (F^1)^{st} CH^p(A \times_k A; \mathbb{Q}) \subseteq (F^1)^{sr} CH^p(A \times_k A; \mathbb{Q})
$$

as desired.
References


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