Homology of zero-divisors

Reza Akhtar and Lucas Lee

Abstract

Let $R$ be a commutative ring with unity. We define a semisimplicial abelian group based on the structure of the semigroup of ideals of $R$ and investigate various properties of the homology groups of the associated chain complex.

1 Introduction

Let $R$ be a commutative ring with unity. The set $Z(R)$ of zero-divisors in a ring does not possess any obvious algebraic structure; consequently, the study of this set has often involved techniques and ideas from outside algebra. Several recent attempts, among them [2], [3] have focused on studying the so-called zero-divisor graph $\Gamma_R$, whose vertices are the zero-divisors of $R$, with $xy$ being an edge if and only if $xy = 0$. This object $\Gamma_R$ is somewhat unwieldy in that it has many symmetries; for example, if $u \in R^*$ is any unit, then $x \mapsto ux$ induces a (graph) automorphism of $\Gamma_R$. One way of treating this issue – following an idea of Lauve [5] – is to work with the ideal zero-divisor graph $I_R$. In effect, one replaces zero-divisors of $R$ by proper ideals with nonzero annihilator; this is the approach adopted by the authors in [1]. Such a perspective also has its shortcomings; for instance, it does not adequately detect the phenomenon of there being three distinct proper ideals $I, J, K$ in $R$ with $IJK = 0$, but $IJ \neq 0$, $IK \neq 0$, $JK \neq 0$.

In this paper we adopt a different philosophy, using a new type of homology to study $Z(R)$ and capture the situation described above. Roughly speaking, if we denote by $Z_n(R)$ the free abelian group generated by the set of $(n+1)$-tuples $(I_0, \ldots, I_n)$ of distinct ideals of $R$ such that $I_0 \cdot \ldots \cdot I_n \neq 0$, there are obvious maps $Z_n(R) \rightarrow Z_{n-1}(R)$ obtained by forgetting one of the factors. This gives $Z(R)$ the structure of a semisimplicial abelian group; hence we may speak of its associated chain complex $C(R)$. Our homology groups $H_s(R)$ are then defined as the homology groups of a certain quotient of $C(R)$. The idea behind this construction was sketched by Lauve in [5], although the precise definition is due to the authors.
After giving a precise definition of these homology groups $H_\ast(R)$, we study the group $H_0(R)$ in depth and compute $H_1(\mathbb{Z}/p^r\mathbb{Z})$ when $p$ is a prime and $r \geq 1$ is an integer. We then give some conditions on $R$ sufficient to ensure that $H_n(R) = 0$ for $n > 0$. In the last section we consider the Euler characteristic $\chi(R) = \sum_{n=0}^{\infty} (-1)^n \text{rk} H_n(R)$. Using some ideas from partition theory, we prove the surprising result that $\chi(\mathbb{Z}/p^r\mathbb{Z})$ is always either 0, 1, or 2, depending on the value of $r$ relative to the “pentagonal” numbers $m(3m-1)/2$ and the related numbers $m(3m+1)/2$. We also derive formulas for the Euler characteristic for some other special types of finite rings.

The authors would like to thank Aaron Lauve, David Anderson, Dennis Keeler, Neil O. Smith, Siamak Yassemi and Tao Jiang for useful conversations and references. They are also deeply grateful to Dan Pritikin for his invaluable insight in elucidating the combinatorics of the calculation of the Euler characteristic in Theorem 6.1. Finally, they thank Miami University for financial support of the second author’s research during the summer of 2003.

2 Preliminaries

Let $R$ be a commutative ring and $\mathcal{P}$ the set of proper ideals of $R$. For each $n \geq 0$, let $S_n(R)$ be the set of ordered $(n+1)$-tuples $(I_0, \ldots, I_n)$, where $I_0, \ldots, I_n$ are distinct proper ideals of $R$ and $I_0I_1 \cdots I_n \neq 0$; let $S_{-1}(R)$ be a set consisting of one element. If there is no danger of ambiguity, we simply write $S_n$ instead of $S_n(R)$. Observe that for each $i$, $0 \leq i \leq n$, there is a “face map” $\phi^n_i : S_n \to S_{n-1}$ defined by $\phi^n_i(I_0, \ldots, I_n) = (I_0, \ldots, \hat{I}_i, \ldots, I_n)$. Moreover, $S_0(R) = \emptyset$ if and only if $R$ is a field, so when $R$ is not a field, there is a unique “augmentation” map $\varepsilon : S_0(R) \to S_{-1}(R)$. Now for each $n \geq -1$, let $Z_n$ be the free abelian group generated by $S_n$. We denote by $[I_0, \ldots, I_n]$ the basis element corresponding to $(I_0, \ldots, I_n) \in S_n$. Likewise, the various face maps $\phi^n_i$ extend $\mathbb{Z}$-linearly to maps $\phi_i^n : Z_n \to Z_{n-1}$; moreover, if $S_0 \neq \emptyset$, there is a unique $\mathbb{Z}$-linear map $\varepsilon : Z_0 \to Z_{-1} = \mathbb{Z}$ defined by $\varepsilon(\sum n_i(I_i)) = \sum n_i$. Thus, there is a semisimplicial abelian group:

$$Z \left( R \right): \quad \ldots \implies Z_1 \implies Z_0$$

with augmentation $\varepsilon : Z_0 \to \mathbb{Z}$ if $R$ is not a field.

This in turn gives rise to an (augmented) chain complex in the standard manner by taking an alternating sum of face maps. For each $n \geq 0$, define $\delta_n = \sum_{i=0}^{n} (-1)^i \phi_i^n$; then we have a complex:
$C_i(R) : \ldots \delta_i \rightarrow Z_i \delta_0 \rightarrow Z_0$ of abelian groups.

In practice, the $Z_n$ are too large to be useful invariants; in particular, we chose $Z_n$ to be the free $\mathbb{Z}$-module with basis $S_n$, which consisted of ordered $(n+1)$-tuples of ideals of $R$ having nonzero product. Because multiplication in $R$ is commutative, the order of the ideals in this $(n+1)$-tuple ought not to matter; it might appear more natural to work with unordered $(n+1)$-tuples. Unfortunately, the definition of the face maps does depend on the ordering within each such tuple, so we resort instead to the following device: for each $n \geq 0$, let $R_n$ denote the subgroup of $Z_n$ generated by elements of the form:

$$[I_0, \ldots, I_n] - (-1)^{sgn} \sigma [I_{\sigma(0)}, \ldots, I_{\sigma(n)}]$$

where $\sigma$ is an element of the symmetric group $S_{n+1}$ (viewed as permutations of the set $\{0, \ldots, n\}$) and $[I_0, \ldots, I_n]$ is a basis element of $Z_n$. Set $T_n = Z_n/R_n$.

We claim that $\delta_n(R_n) \subseteq R_{n-1}$. Thus we must show

$$\delta_n([I_0, \ldots, I_n]) \equiv (-1)^{sgn} \sigma \delta_n([I_{\sigma(0)}, \ldots, I_{\sigma(n)}]) \pmod{R_{n-1}}.$$ 

Since every permutation may be written as a product of transpositions, we may reduce to the case that $\sigma$ is the transposition which exchanges $r$ and $s$, where $0 \leq r < s \leq n$. In this case,

$$(-1)^{sgn} \sigma \delta_n([I_{\sigma(0)}, \ldots, I_{\sigma(n)}]) = - \sum_{i=0}^{n} (-1)^i [I_{\sigma(0)}, \ldots, \hat{I}_{\sigma(i)}, \ldots, I_{\sigma(n)}]$$

$$= \sum_{i \neq r,s} (-1)^{i+1} [I_0, \ldots, I_{r-1}, I_s, I_{r+1}, \ldots, \hat{I}_{i}, \ldots, I_{s-1}, I_r, I_{s+1}, \ldots, I_n]$$

$$+ (-1)^{r+1} [I_0, \ldots, I_{r-1}, I_{r+1}, \ldots, I_{s-1}, I_r, I_{s+1}, \ldots, I_n]$$

$$+ (-1)^{s+1} [I_0, \ldots, I_{r-1}, I_s, I_{r+1}, \ldots, I_{s-1}, I_{s+1}, \ldots, I_n]$$

$$\equiv \sum_{i \neq r,s} (-1)^i [I_0, \ldots, I_{r-1}, I_r, I_{r+1}, \ldots, \hat{I}_{i}, \ldots, I_{s-1}, I_s, I_{s+1}, \ldots, I_n]$$

$$+ (-1)^{s} [I_0, \ldots, I_{r-1}, I_r, I_{r+1}, \ldots, I_{s-1}, I_s, I_{s+1}, \ldots, I_n]$$

$$+ (-1)^{2s-r} [I_0, \ldots, I_{r-1}, I_{r+1}, \ldots, I_{s-1}, I_s, I_{s+1}, \ldots, I_n] \pmod{R_{n-1}}$$

3
\[\equiv \sum_{i=0}^{n} (-1)^i [I_0, \ldots, \hat{I}_i, \ldots, I_n](\text{mod } R_{n-1}) \equiv \delta_n([I_0, \ldots, I_n])(\text{mod } R_{n-1})\]

Thus \(\delta_n(R_n) \subseteq R_{n-1}\) for all \(n \geq 1\), and hence \(C_*(R)\) factors through a complex:

\[\tilde{C}_*(R) : \ldots \xrightarrow{\partial_1} T_1 \xrightarrow{\partial_0} T_0 \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0\]

By abuse of notation, we continue to use the symbol \([I_0, \ldots, I_n]\) to denote the class of \([I_0, \ldots, I_n]\) in \(T_n\); hence the formula for \(\partial_n\) (on generators) reads: \(\partial_n([I_0, \ldots, I_n]) = \sum_{i=0}^{n} (-1)^i [I_0, \ldots, \hat{I}_i, \ldots, I_n]\).

Finally we define the homology groups:

\[H_n(R) = \begin{cases} 
\ker (\partial_{n-1}) & \text{if } n > 0 \\
\frac{\ker (\partial_n)}{\im (\partial_n)} & \text{if } n = 0
\end{cases}\]

If \(\text{rk } H_n(R)\) is finite for all \(n\) and zero for sufficiently large \(n\), we define the Euler characteristic of \(R\):

\[\chi(R) = \sum_{n=0}^{\infty} (-1)^n \text{rk } H_n(R)\]

Since a field has no proper ideals, we immediately have:

**Proposition 2.1.** Let \(F\) be a field. Then \(H_n(F) = 0\) for all \(n \geq 0\).

The term “homology” is used somewhat loosely, since neither the complexes \(\tilde{C}_*(R)\) nor the groups \(H_n(R)\) are functorial in \(R\). This is not particularly surprising: given a ring homomorphism \(f : R \rightarrow S\), if \([I_0, \ldots, I_n] \in T_n(R)\), it is possible that \(I_0 \ldots I_n = 0\) or one of the \(f(I_i)\) may be zero, so it does not necessarily follow that \([f(I_0), \ldots, f(I_n)]\) makes sense as an element of \(T_n(S)\). Similarly, if \([J_0, \ldots, J_n] \in T_n(S)\), it does not follow that \([f^{-1}(J_0), \ldots, f^{-1}(J_n)]\) defines an element of \(T_n(R)\).

The following well-known device is often useful in computing the Euler characteristic:

**Proposition 2.2.** Suppose \(\text{rk } T_n\) is finite for all \(n\) and \(T_n = 0\) for \(n \gg 0\). Then

\[\chi(R) = \sum_{n=0}^{\infty} (-1)^n \text{rk } T_n\]
Proof.
By definition of $H_0(R)$, there is an exact sequence:

$$0 \to \text{Im } \partial_0 \to T_0 \to H_0(R) \to 0$$

and for each $n \geq 1$, there is a short exact sequence:

$$0 \to \text{Im } \partial_n \to \text{Ker } \partial_{n-1} \to H_n(R) \to 0$$

Since the rank is additive across exact sequences, we have:

$$\chi(R) = \sum_{n=0}^{\infty} (-1)^n \text{rk } H_n = \text{rk } T_0 - \text{rk } \text{Im } \partial_0 + \sum_{n=1}^{\infty} (-1)^n (\text{rk } \text{Ker } \partial_{n-1} - \text{rk } \text{Im } \partial_n)$$

Furthermore for any $n \geq 0$, $\text{rk } \text{Im } \partial_n = \text{rk } T_{n+1} - \text{rk } \text{Ker } \partial_n$, so the above expression for $\chi(R)$ becomes:

$$\chi(R) = \text{rk } T_0 - \text{rk } T_1 + \text{rk } \text{Ker } (\partial_0) + \sum_{n=1}^{\infty} (-1)^n (\text{rk } \text{Ker } \partial_{n-1} - \text{rk } T_{n+1} + \text{rk } \text{Ker } \partial_n)$$

$$= \text{rk } T_0 - \text{rk } T_1 + \sum_{n=1}^{\infty} (-1)^n \text{rk } T_{n+1} = \sum_{n=0}^{\infty} (-1)^n \text{rk } T_n$$

3 The group $H_0(R)$

Let $R$ be a commutative ring with unity. In order to analyze $H_0(R)$, we recall the construction of the so-called ideal graph $\mathcal{I}_R$. This is a (simple) graph whose vertices are the proper ideals of $R$, with $\{I, J\}$ being an edge if and only if $IJ = 0$. We will be more interested in the complement graph $\overline{\mathcal{I}_R}$, whose vertices are the same as $\mathcal{I}_R$, but in which $\{I, J\}$ is an edge if and only if $IJ \neq 0$.

If $\sum_{i=1}^{n} [I_i] \in T_0$ is an element whose class in $H_0(R)$ is zero, this means that $\sum_{i=1}^{n} [I_i] = \partial_0(\sum_{j=1}^{n} c_j [A_j, B_j])$ for some integers $c_j$ and proper ideals $A_j, B_j$. Without loss of generality we may assume $c_j = \pm 1$. Equality still holds if we replace $[A_j, B_j]$ by $-[B_j, A_j]$, so we may always write $\sum_{i=1}^{n} [I_i] = \partial_0(\sum_{k=1}^{r} [C_k, D_k])$ for some proper ideals $C_k, D_k$.  

5
Proposition 3.1. Let $I$ and $J$ be distinct proper ideals of $R$. Then $[I]$ and $[J]$ have the same class in $H_0(R)$ if and only if $I$ and $J$ lie in the same connected component of the graph $\overline{I}_R$.

Proof.
If $I$ and $J$ are in the same connected component of $\overline{I}_R$, then there is some path $I = A_0 - A_1 - \ldots - A_n = J$ connecting $I$ and $J$, where the $A_i$ are ideals such that for each $i = 0, \ldots, n-1$, $A_iA_{i+1} \neq 0$. This directly implies that $\sum_{i=0}^{n-1} [A_i, A_{i+1}]$ is an element of $T_1$, and by direct calculation we see that

$$\partial_0(\sum_{i=0}^{n-1} [A_i, A_{i+1}]) = [A_0] - [A_n] = [I] - [J].$$

Hence $[I] = [J]$ in $H_0(R)$.

Conversely, suppose $[I]$ and $[J]$ define the same class in $H_0(R)$. Then $[I] - [J] = \partial_0(\sum_{i=0}^{n-1} [A_i, B_i]) = \sum_{i=0}^{n-1} [A_i] - [B_i]$ where $A_i, B_i$ are distinct proper ideals of $R$ and $A_iB_i \neq \emptyset$. Let $n$ be the smallest integer for which this is possible. We prove by induction on $n$ that, after suitable reordering of the $A_i$ and $B_i$, there is a path in $\overline{I}_R$ from $I$ to $J$.

We may assume without loss of generality that $A_0 = I$ and $B_n = J$. If $B_0 = J$, then $IJ \neq 0$ and we are done. Otherwise, assume $B_0 \neq J$; that is, $n > 0$. Since

$$[I] - [J] = [I] - [B_0] + [A_1] - [B_1] + \ldots + [A_n] - [B_n]$$

is a relation in a free abelian group, we may assume without loss of generality that $A_1 = B_0$. Then, adding $[B_0] - [I]$ to both sides of this equation, we get

$$[B_0] - [J] = [A_1] - [B_1] + \ldots + [A_n] - [B_n]$$

so by induction there is a path in $\overline{I}_R$ from $B_0$ to $J$. Since $A_0B_0 \neq 0$, this means that $\{A_0, B_0\}$ is an edge in $\overline{I}_R$, and hence that there is a path from $A_0 = I$ to $J$.

Proposition 3.2. Let $I_1, \ldots, I_n$ be distinct proper ideals of $R$ lying in mutually distinct connected components of $\overline{I}_R$. Then the classes of $[I_1], \ldots, [I_n]$ are linearly independent in $H_0(R)$.

Proof.
If $R$ is a field, the assertion is trivial. Otherwise, let $C_1, \ldots, C_r$ be the components of $\overline{I}_R$. Suppose the class of $\sum_{i=1}^n c_i[I_i]$ in $H_0(R)$ is 0. We may assume that each $I_i$ lies in component $C_i$ of $\overline{I}_R$. Now
\[ \sum_{i=1}^{n} c_i[I_i] = \partial_0(\sum_{j=1}^{m} [A_j, B_j]) \]

for some distinct proper ideals \( A_j, B_j \) such that \( A_jB_j \neq 0 \). Since \([A_j, B_j] \in T_1\), \( A_j \) and \( B_j \) must lie in the same component of \( \mathcal{I}_R \). For each \( k, 1 \leq k \leq r \), let \( J_k = \{ j : 1 \leq j \leq m : A_j \in C_k \} \). Then it follows from the above equation that

\[ c_k[I_k] = \partial_0(\sum_{j \in J_k} [A_j] - [B_j]) \]

Applying \( \varepsilon \) to both sides of this equation, we have \( c_k = 0 \) for all \( k \).

Combining the previous two propositions, we have:

**Corollary 3.3.** Let \( R \) be a ring, and \( r \) the number of connected components of \( \mathcal{I}_R \). Then

\[ H_0(R) \cong \mathbb{Z}^r. \]

Corollary 3.3 is a useful tool for calculating \( H_0(R) \) in particular cases; nevertheless, using only elementary facts about ideals, one can prove even more. We begin with an elementary lemma:

**Lemma 3.4.** Let \( R \) be a ring and \( m_1, m_2 \) distinct maximal ideals of \( R \). If \( m_1m_2 = 0 \), then \( R \) is isomorphic to a product of two fields.

**Proof.**
Let \( p \) be a prime ideal of \( R \). Then \( p \supseteq m_1m_2 = 0 \), so \( p \supseteq m_1 \) or \( p \supseteq m_2 \); i.e. \( p = m_1 \) or \( p = m_2 \). Hence \( m_1 \) and \( m_2 \) are the only prime ideals of \( R \) and so \( R \) is an Artin ring with two maximal ideals. By the structure theorem for Artin rings, \( R \cong R_1 \times R_2 \), where \( R_1, R_2 \) are Artin local rings with respective maximal ideals \( n_1, n_2 \). Then without loss of generality, \( m_1 = n_1 \times R_2 \) and \( m_2 = R_1 \times n_2 \). Thus, \( 0 = m_1m_2 = n_1 \times n_2 \) so \( n_1 = 0, n_2 = 0 \) and so \( R_1, R_2 \) are fields.

**Proposition 3.5.** Let \( R \) be a nonlocal ring which is not isomorphic to the product of two fields. Then \( H_0(R) \cong \mathbb{Z} \).

**Proof.**
By Corollary 3.3 it suffices to prove that \( \mathcal{I}_R \) is connected. Indeed, let \( m_1, m_2 \) be distinct maximal ideals of \( R \). If \( I \) is any other proper ideal of \( R \), then \( \text{ann}(I) \) is a proper ideal of \( R \), so \( \text{ann}(I) \) does not contain both \( m_1 \) and \( m_2 \). Hence for each such \( I \), at least one of \( \{ I, m_1 \}, \{ I, m_2 \} \) is an edge in \( \mathcal{I}_R \). If \( m_1m_2 = 0 \), then it follows from
Lemma 3.4 that $R$ is isomorphic to a product of two fields. Thus $m_1m_2 \neq 0$, \{m_1, m_2\} is an edge of $\mathcal{I}_R$, and it follows that $\mathcal{I}_R$ is connected.

We have seen that $H_0(F) = 0$ when $F$ is a field and $H_0(R) \cong \mathbb{Z}$ for a large class of rings. Direct computation shows that if $F_1$ and $F_2$ are fields, then $H_0(F_1 \times F_2) \cong \mathbb{Z}^2$ and $H_n(F_1 \times F_2) = 0$ for all $n > 0$. A natural question that arises is: given any integer $s \geq 0$, is there a ring $R$ such that $H_0(R) \cong \mathbb{Z}^s$? The discussion above shows that when $s \geq 3$, any such $R$ must necessarily be local. Following an idea supplied to us by Dennis Keeler, we show below that the rank of $H_0(R)$ may be arbitrarily large.

Let $k$ be a field and $x_1, \ldots, x_s$ independent indeterminates. Let $S$ be the localization of $k[x_1, \ldots, x_s]$ with respect to the maximal ideal $(x_1, \ldots, x_s)$. Now let $I$ be the ideal of $k[x_1, \ldots, x_s]$ generated by all products $x_i x_j$, where $i \leq j$. Since $I \subseteq (x_1, \ldots, x_s)$, $I$ corresponds, in the usual manner, to an ideal $\bar{I} \subseteq S$. Now let $R = S/\bar{I}$. Observe now that the proper ideals of $R$ correspond bijectively to ideals $(x_{i_1}, \ldots, x_{i_\nu}) \subseteq k[x_1, \ldots, x_s]$, where $1 \leq \nu \leq s$ and $1 \leq i_1 < \ldots < i_\nu \leq s$. Furthermore, each such ideal (of $R$), when multiplied by any other, yields 0. Thus $\mathcal{I}_R$ is a completely disconnected graph on $2^s - 2$ vertices, and so $H_0(R) \cong \mathbb{Z}^{2^s-2}$.

4 Calculation of $H_1(\mathbb{Z}/p^r\mathbb{Z})$

In this section, we compute the group $H_1(\mathbb{Z}/p^r\mathbb{Z})$ where $p$ is a prime number and $r \geq 1$ an integer. It is easy to see by direct calculation that if $r \leq 3$, then $H_1(\mathbb{Z}/p^r\mathbb{Z}) = 0$. We assume henceforth that $r \geq 4$.

Recall first that

$$H_1(R) = \frac{\ker (\partial_0 : T_1 \to T_0)}{\text{im} (\partial_1 : T_2 \to T_1)}$$

where

$$\partial_0(\sum_j [A_j, B_j]) = \sum_j [A_j] - [B_j]$$

and

$$\partial_1(\sum_j [A_j, B_j, C_j]) = \sum_j [B_j, C_j] - \sum_j [A_j, C_j] + \sum_j [A_j, B_j]$$
Definition 4.1. Let $n \geq 0$ be an integer. An element $\alpha \in T_1$ is called an $n$-circuit (or simply a circuit) if there exist proper ideals $I_1, \ldots, I_n$ of $R$ such that

$$\alpha = [I_1, I_2] + \ldots + [I_{n-1}, I_n] + [I_n, I_1]$$

A 3-circuit is called a triangle.

Clearly the definition has been chosen to reflect the fact that in the above context, $I_1 - I_2 - \ldots I_n - I_1$ is a circuit in the graph $\bar{I}_Z/p^rZ$. The analysis of $\text{Ker } \partial_0$ proceeds by a sequence of lemmas.

Lemma 4.2. Every element $\beta \in \text{Ker } \partial_0$ may be written

$$\beta = \sum_{k=1}^{m} \alpha_k$$

where each $\alpha_k$ is a circuit.

Proof. The proof is by induction on the number of symbols in $\beta$. If $\beta = 0$, the claim is clear. Otherwise, let $\beta = \sum_{j=1}^{r} [A_j, B_j]$ with $r$ chosen to be as small as possible. We may assume that there is no pair of integers $(j_1, j_2)$, $1 \leq j_1 < j_2 \leq r$ such that $A_{j_1} = B_{j_2}$ and $A_{j_2} = B_{j_1}$, for then we may use the relation $[I, J] = -[J, I]$ in $T_1$ to simplify the expression for $\beta$ and obtain a relation with smaller $r$.

Since $\beta \in \text{Ker } \partial_0$, we have:

$$0 = \partial_0(\beta) = \partial_0(\sum_{j=1}^{r} [A_j, B_j]) = \sum_{j=1}^{r} [A_j] - [B_j]$$

Since this is a relation in the (free abelian) group $T_0$, it follows that there is some $j$ such that $B_1 = A_j$. Without loss of generality we may assume that $j = 2$. By the previous discussion, it follows that $A_1 \neq B_2$. Now it must be the case that there is some $j$ such that $B_2 = A_j$; without loss of generality, we assume that $j = 3$. Continue this procedure until one reaches $s \leq r$ such that $B_s = A_1$. Then

$$\beta_1 = [A_1, B_1] + [B_1, B_2] + \ldots + [B_{s-2}, B_{s-1}] + [B_{s-1}, A_1]$$

is a circuit in $T_1$. By induction, $\beta - \beta_1$ is a sum of circuits in $T_1$; hence $\beta$ itself is a sum of circuits.

Lemma 4.3. Every nonzero circuit in $T_1 = T_1(\mathbb{Z}/p^r\mathbb{Z})$ may be written as a sum of triangles.
Proof.
Let $\alpha = \sum_{j=1}^{r-1}[A_j, A_{j+1}] + [A_r, A_1]$ be a circuit in $T_1$. If $\alpha$ is a 3-circuit, there is nothing to prove. By induction, it suffices to prove that $\alpha$ has a chord, i.e. there exist distinct integers $i, j, 1 \leq i < j \leq r$ such that $[A_i, A_j] \in T_1$ and $j - i > 1$. Suppose $\alpha$ is an $n$-circuit, with $n > 3$. For each $k$, $1 \leq k \leq r - 1$, let $I_k$ denote the ideal of $\mathbb{Z}/p^r\mathbb{Z}$ generated by (the class of) $p^k$. Let $\mathcal{S} = \{I_k : 1 \leq k < r/2\}$. Observe that if $C, D \in \mathcal{S}$, then $[C, D] \in T_1$. Furthermore, if $[C, D] \in T_1$ and $C \notin \mathcal{S}$, then $D$ must be in $\mathcal{S}$.

If all the $A_i$ appearing in the cycle $\alpha$ are members of $\mathcal{S}$, then by the above observation $[A_1, A_2] + [A_2, A_3] + [A_3, A_1]$ is a triangle. If not, then we may assume without loss of generality that $A_2 \notin \mathcal{S}$. Since $[A_1, A_2] \in T_1$ and $[A_2, A_3] \in T_1$, we must have $A_1 \in \mathcal{S}$, $A_3 \in \mathcal{S}$. This forces $[A_1, A_3] \in T_1$, which completes the proof.

Lemma 4.4. Every triangle in $T_1(\mathbb{Z}/p^r\mathbb{Z})$ may be written as a sum of triangles of the form $\tau_{ij} = [I_1, I_i] + [I_i, I_j] + [I_j, I_1]$, where $1 < i, j < r$.

Proof.
This follows immediately from the formal identity:

$[I_h, I_i] + [I_i, I_j] + [I_j, I_h] =$

$([I_1, I_h] + [I_h, I_i] + [I_i, I_1]) + ([I_1, I_i] + [I_i, I_j] + [I_j, I_1]) + ([I_1, I_j] + [I_j, I_h] + [I_h, I_1])$

$= \tau_{hi} + \tau_{ij} + \tau_{jh}$

Lemma 4.5. The set of triangles $\mathcal{T} = \{\tau_{ij} : 1 < i < j < r\}$ is $(\mathbb{Z})$-linearly independent in $T_1$.

Proof.
This follows readily from the fact that $\tau_{ij}$ is the only member of $\mathcal{T}$ involving the symbol $[I_i, I_j]$.

It follows from the sequence of lemmas above that:

Corollary 4.6. The group $\text{Ker } \partial_0$ is a free abelian group with basis $\mathcal{T}$.

In fact, $\tau_{ij} \in \mathcal{T}$ if and only if $i + j < r$, so an elementary counting argument gives:
Corollary 4.7. The rank of Ker $\partial_0$ is $\frac{(r-4)^2}{4}$ if $r$ is even or $\frac{(r-4)^2-1}{4}$ if $r$ is odd.

We now examine the group $\text{Im } \partial_1$. Observe that:

$$\gamma = \partial_1([I_i, I_j, I_k]) = [I_i, I_j] - [I_i, I_k] + [I_j, I_k] = [I_i, I_j] + [I_j, I_k] + [I_k, I_i]$$

is a triangle of $T_1$.

Since $I_iI_jI_k \neq 0$ and $I_1$ contains $I_i$, $I_j$ and $I_k$, it follows readily that each of the symbols $[I_1, I_i, I_j]$, $[I_1, I_i, I_k]$ and $[I_1, I_j, I_k]$ are in $T_2$; furthermore,

$$\gamma = \partial_1([I_i, I_j, I_k]) = \partial_1([I_1, I_i, I_j]) + \partial_1([I_1, I_j, I_k]) + \partial_1([I_1, I_k, I_i]) = \tau_{ij} + \tau_{jk} + \tau_{ki}$$

so in fact $\text{Im } \partial_1$ is generated by those elements $\tau_{ij} \in T$ such that $1 + i + j < r$, i.e. $i + j < r - 1$.

By the same computation as used to derive Corollary 4.7, we obtain:

Corollary 4.8. The group $\text{Im } \partial_1$ is a free abelian group of rank

$$\frac{(r-5)^2-1}{4} \quad \text{if } r \text{ is even or } \frac{(r-5)^2}{4} \quad \text{if } r \text{ is odd.}$$

In particular, we observe that the basis elements $\tau_{ij}$ for $\text{Im } (\partial_1)$ identified in the previous discussion are a subset of those identified as a basis for Ker $(\partial_0)$. Thus, we have:

Corollary 4.9. Suppose $r \geq 4$. Then $H_1(\mathbb{Z}/p^r \mathbb{Z})$ is a free abelian group of rank $\frac{r-4}{2}$ if $r$ is even or $\frac{r-5}{2}$ if $r$ is odd.

5 Acyclicity

In this section, we make a general study of the higher homology groups $H_n(R)$, $n > 0$; in particular, we give various conditions sufficient for these groups to be zero.

Towards this end, it is convenient to introduce some notation: if $I_j_0, \ldots, I_j_m$ ($j = 1 \ldots r$) and $J_0, \ldots, J_n$ are mutually distinct ideals of a ring $R$ such that $[I_j_0, \ldots, I_j_m] \in T_m(R)$ for each $j$ and $[J_0, \ldots, J_n] \in T_n(R)$, and also $I_j_0 \ldots I_j_m, J_0 \ldots J_n \neq 0$, for each $j$, we write:

$$\sum_{j=1}^{r} [I_j_0, \ldots, I_j_m] \times [J_0, \ldots, J_n] = \sum_{j=1}^{r} [I_j_0, \ldots, I_j_m, J_0, \ldots, J_n]$$

11
Lemma 5.1. (Acyclicity Lemma) Suppose $n > 0$ and $\alpha = \sum_{j=1}^{r} [I_{j_0}, \ldots, I_{j_n}] \in \text{Ker} (\partial_{n-1})$. If there exists an ideal $J \not\in \{I_{j_k} : 1 \leq j \leq r, 0 \leq k \leq n\}$ such that $JI_{j_0} \ldots I_{j_n} \neq 0$ for all $j, 1 \leq j \leq r$, then $\alpha \in \text{Im} (\partial_n)$. Thus the class of $\alpha$ in $H_n(R)$ is zero.

Proof.
If such $J$ exists, then

$$\partial_n((-1)^{n+1} \sum_{j=1}^{r} [I_{j_0}, \ldots, I_{j_n}] \times [J]) = (-1)^{n+1} \sum_{i=0}^{r} \sum_{j=1}^{r} (-1)^{n}[I_{j_0}, \ldots, \hat{I}_{j_i}, \ldots, I_{j_n}, J] + \alpha$$

$$= -\partial_{n-1}(\alpha) \times [J] + \alpha = \alpha$$

So indeed $\alpha \in \text{Im} (\partial_n)$, as desired.

Theorem 5.2. Let $R$ be a ring satisfying at least one of the following conditions:

- There exists a nonzero element $x \in R$ which is neither a unit nor a zero-divisor.
- $R$ has infinitely many maximal ideals.
- $R$ is reduced, Noetherian, and of positive (Krull) dimension.

Then $H_n(R) = 0$ for all $n > 0$.

Proof.
First, suppose $x \in R$ is a nonzero element which is neither a unit nor a zero-divisor. Then it is easy to see that $x^i$ and $x^j$ are associate if and only if $i = j$. Thus,

$$(x) \supset (x^2) \supset (x^3) \supset \ldots$$

is a descending chain of distinct ideals. Furthermore, if $I$ is a nonzero ideal, then $(x^i)I \neq 0$, for any $i \geq 1$ because $x$ (and hence $x^i$) is not a zero-divisor. Given any $n > 0$ and $\alpha = \sum_{j=1}^{r} [I_{j_0}, \ldots, I_{j_n}] \in \text{Ker} (\partial_{n-1})$ as in Lemma 5.1, choose $m$ such that $(x^m) \neq I_{j_k}$ for all $j, k$. Then $J = (x^m)$ satisfies the hypotheses of the Lemma and the assertion follows.

Now suppose $R$ has infinitely many maximal ideals, and suppose $\alpha$ is as above. For each $j$, let $A_j = \text{ann}(I_{j_0} \ldots I_{j_n})$; $A_j$ is a proper ideal of $R$, so choose some maximal ideal $m_j$ such that $A_j \subseteq m_j$. For each $j, 1 \leq j \leq r$ and $k, 1 \leq k \leq n$, choose a maximal ideal $m_{jk}$ such that $I_{j_k} \subseteq m_{jk}$. Now let

$$D = \bigcup_{j=1}^{r} m_j \cup \bigcup_{j=1}^{r} \bigcup_{k=1}^{n} m_{jk}$$

12
Let \( \mathfrak{m} \) be some other maximal ideal of \( R \) not equal to any \( \mathfrak{m}_j \) or \( \mathfrak{m}_{jk} \). By \([4]\), Proposition 1.11, \( \mathfrak{m} \not\subseteq D \). Evidently, \((x)\) is a proper ideal of \( R \). Furthermore, since \( x \not\in \mathfrak{m}_{jk} \), \((x) \neq I_{jk}\) for any \( j, k \). Finally, \( x \not\in \mathfrak{m}_j \supseteq A_j \) implies that \((x)I_{j_0}\ldots I_{j_n} \neq 0\) for all \( j \). Thus, \( J = (x) \) satisfies the hypotheses of Lemma 5.1, and the assertion is proved.

Last, suppose \( R \) is reduced, Noetherian, and \( \dim R > 0 \). Let \( \mathfrak{p}_0 \) be a minimal prime ideal of \( R \) which is not also maximal. Then \( \dim(R/\mathfrak{p}_0) > 0 \), so in particular \( R/\mathfrak{p}_0 \) is not Artinian. Thus, there is a strictly descending sequence of ideals of \( R \):

\[
R \supseteq J_1 \supseteq J_2 \supseteq \ldots
\]
each of which strictly contains \( \mathfrak{p}_0 \).

Let \( \mathfrak{p}_0, \ldots, \mathfrak{p}_n \) be the minimal prime ideals of \( R \); there are only finitely many of them because \( R \) is Noetherian (\([4]\), Chapter 6, Exercise 9). It is well-known (cf. \([4]\), Prop. 1.8) that the nilradical of \( R \) is the intersection of the prime ideals of \( R \) – hence also of the minimal prime ideals of \( R \). Thus in our case, \( \bigcap_{i=0}^{n} \mathfrak{p}_i = 0 \).

We claim that \( IJ_m \neq 0 \) for any nonzero ideal \( I \) and any \( m \geq 1 \). Suppose to the contrary that \( IJ_m = 0 \). Since \( \bigcap_{i=0}^{n} \mathfrak{p}_i = 0 \), this means \( \mathfrak{p}_i \supseteq IJ_m \) for each \( i \). Since \( \mathfrak{p}_i \) is prime, \( \mathfrak{p}_i \supseteq I \) or \( \mathfrak{p}_i \supseteq J_m \). In the latter case, \( \mathfrak{p}_i \supseteq J_m \supseteq \mathfrak{p}_0 \), so by minimality of \( \mathfrak{p}_i \), we must have \( \mathfrak{p}_i = J_m = \mathfrak{p}_0 \). However, \( J_m \) strictly contains \( \mathfrak{p}_0 \), so this is impossible. Thus, we must have \( \mathfrak{p}_i \supseteq I \) for each \( i \); hence, \( 0 = \bigcap_{i=0}^{n} \mathfrak{p}_i \supseteq I \) and so \( I = 0 \).

Continuing with the proof of Theorem 5.2, suppose \( n > 0 \) and \( \alpha = \sum_{j=1}^{r}[I_{j_0}, \ldots, I_{j_n}] \in \text{Ker } (\partial_n) \) as in Lemma 5.1. Choose \( m \geq 1 \) such that \( J_m \not\subseteq \{I_{j_k} : 1 \leq j \leq r, 0 \leq k \leq n\} \). Then the previous paragraph shows that for any \( j, 1 \leq j \leq r, JI_{j_0}\ldots I_{j_n} \neq 0 \); thus we may take \( J = J_m \) and apply Lemma 5.1 to conclude.

6 \( \chi \) for finite rings

Theorem 5.2 establishes that the higher homology groups are uninteresting for a large class of rings. Finite rings, on the other hand, satisfy none of the conditions of the theorem; in this section, we examine these rings more closely. While the prospect of computing the actual homology groups seems daunting, the Euler characteristic turns out to be a much more tractable object. In particular, if \( R \) is a finite ring – hence having only finitely many ideals – it is clear from the definition that each \( T_n(R) \)
has finite rank and that $T_n(R) = 0$ for sufficiently large $n$. Hence the hypotheses of Proposition 2.2 are satisfied and we may use it to compute the Euler characteristic. In particular, let $U_n = U_n(R)$ denote the number of unordered $(n+1)$-tuples $\{I_0, \ldots, I_n\}$ of distinct ideals whose product is nonzero. Then we have the convenient formula
\[
\chi(R) = \sum_{n=0}^{\infty} (-1)^n |U_n|
\]
Throughout this section, if a set is denoted by an uppercase letter, we will use the corresponding lower case letter for the number of elements in that set. For example, we will write $u_n$ for $|U_n|$ as defined above.

We begin by examining the same rings encountered in Sec. 4, namely those of the form $R = \mathbb{Z}/p^r\mathbb{Z}$ where $p$ is a prime and $r \geq 1$ is some integer. Recall that for each $i$, $1 \leq i \leq r-1$, there is an ideal $I_i$ of $R$ generated by (the class of) $(p^i)$ and that these are all the proper ideals of $R$. In the following, we implicitly identify the ideal $I_i$ with the integer $i$. Since $U_n$ is the set of unordered $(n+1)$-tuples $\{I_0, \ldots, I_n\}$ of distinct proper ideals of $R$, we have
\[
u_n = \sum_{k=1}^{r-1} P(k, n+1)
\]
where $P(k, n+1)$ represents the number of partitions of $k$ into $(n+1)$ distinct positive integer parts. Hence
\[
\chi(R) = \sum_{n=0}^{\infty} (-1)^n s_n = \sum_{n=0}^{\infty} (-1)^n \sum_{k=1}^{r-1} P(k, n+1) = \sum_{k=1}^{r-1} \sum_{n=1}^{\infty} (-1)^{n+1} P(k, n)
\]
We may interpret the inner sum $\sum_{n=1}^{\infty} (-1)^{n+1} P(k, n) = -\sum_{n=1}^{\infty} (-1)^n P(k, n)$ as the coefficient of $x^k$ in the power series:
\[-(1 - x)(1 - x^2)(1 - x^3) \ldots\]
By Euler’s pentagonal theorem, we have:
\[-(1 - x)(1 - x^2)(1 - x^3) \ldots = -1 + x + x^2 - x^5 - x^7 + x^{12} + x^{15} - x^{22} - x^{26} + \ldots\]
where the pattern of signs on the right (from the second term forth) is $++-\ldots$ and the exponents alternate between the “pentagonal” numbers of the form $P_m = \frac{m(3m-1)}{2}$ and the related numbers $Q_m = \frac{m(3m+1)}{2}$, where $m = 1, 2, 3, \ldots$.\[14\]
Hence
\[ \chi(R) = - \sum_{k=1}^{r-1} \sum_{n=1}^{\infty} (-1)^n P(k, n) \]
is the sum of the coefficients of the terms \( x, x^2, \ldots, x^{r-1} \) appearing in the above series. It is clear from the sign pattern that this sum is either 0, 1, or 2, depending on the value of \( r \) in relation to the numbers \( P_m \) and \( Q_m \).

We summarize our findings in the following:

**Theorem 6.1.** Let \( p \) be a prime and \( r \geq 1 \) an integer. Then \( \chi(\mathbb{Z}/p^r\mathbb{Z}) \) is equal to 0, 1, or 2, depending on the value of \( r \) in relation to the various pentagonal numbers \( \frac{m(3m-1)}{2} \) and the associated numbers \( \frac{m(3m+1)}{2} \).

By being careful with counting methods, we can prove the following theorem, whose proof is facilitated by the paucity of ideals in a field.

**Theorem 6.2.** Let \( R \) be a finite ring and \( F \) a field. Then
\[ \chi(R \times F) = 2 - \chi(R) \]

**Proof.**

Let \( \pi_1, \pi_2 \) denote the projection maps onto the respective factors of \( R \times F \). Recall that for any \( n \geq 0 \), the typical element \( U_n(R \times F) \) is an unordered \((n + 1)\)-tuple \( \{I_0, \ldots, I_n\} \) where \( I_0, \ldots, I_n \neq 0 \). Moreover, each \( I_i = A_i \times B_i \), with \( A_i = \pi_1(I_i) \) being an ideal of \( R \) and \( B_i = \pi_2(I_i) \) an ideal of \( F \), i.e. \( B_i = 0 \) or \( B_i = F \). In order to have \( I_0, \ldots, I_n \neq 0 \), at least one of \( \prod_{i=0}^{n} A_i \neq 0 \) or \( \prod_{i=0}^{n} B_i \neq 0 \). Define:

\[ U^1_n(R \times F) = \{ \{I_0, \ldots, I_n\} \in U_n(R \times F) : \prod_{i=0}^{n} A_i \neq 0 \} \]
\[ U^2_n(R \times F) = \{ \{I_0, \ldots, I_n\} \in U_n(R \times F) : \prod_{i=0}^{n} B_i \neq 0 \} = \{ \{I_0, \ldots, I_n\} \in U_n : B_i = F \text{ for each } i \} \]
\[ U^3_n(R \times F) = U^1_n(R \times F) \cap U^2_n(R \times F) = \{ \{I_0, \ldots, I_n\} \in U_n(R \times F) : B_i = F \text{ for each } i \text{ and } (A_0, \ldots, A_n) \in U_n(R) \} \]

Thus we have \( u_n = u^1_n + u^2_n - u^3_n \).
It is clear from the above description that \( u^3_n(R \times F) = u_n(R) \) and furthermore that if \( \{I_0, \ldots, I_n\} \in U^2_n(R \times F) \), then \( A_0, \ldots, A_n \) are allowed to be any (mutually distinct) proper ideals of \( R \); hence \( u^3_n(R \times F) = \left( \frac{\rho}{n+1} \right) \), where \( \rho \) is the number of proper ideals in \( R \).

The set \( U^1_n \) is slightly more difficult to analyze: define

\[
U^{1,0}_n(R \times F) = \{ \{I_0, \ldots, I_n\} \in U^1_n(R \times F) : I_i \neq R \times 0 \text{ for all } i, 0 \leq i \leq n \}
\]

\[
U^{1,1}_n(R \times F) = U^1_n(R \times F) - U^{1,0}_n(R \times F)
\]

Clearly \( u^{1,0}_n(R \times F) + u^{1,1}_n(R \times F) = u^1_n(R \times F) \). Somewhat more subtly, there is a natural bijective map \( U^{1,0}_n(R \times F) \to U^{1,1}_n(R \times F) \) sending \( \{I_0, \ldots, I_n\} \mapsto \{I_0, \ldots, I_n, R \times 0\} \), so it is also true that \( u^{1,0}_n(R \times F) = u^{1,1}_{n+1}(R \times F) \).

Combining all these relations, we have:

\[
\chi(R \times F) = \sum_{n=0}^{\infty} (-1)^n u_n(R \times F)
\]

\[
= \sum_{n=0}^{\infty} (-1)^n \left( u^1_n(R \times F) + u^2_n(R \times F) - u^3_n(R \times F) \right)
\]

\[
= \sum_{n=0}^{\infty} (-1)^n \left( u^{1,0}_n(R \times F) + u^{1,1}_n(R \times F) + \left( \frac{\rho}{n+1} \right) - u_n(R) \right)
\]

\[
= \sum_{n=0}^{\infty} (-1)^n u^{1,0}_n(R \times F) + \sum_{n=0}^{\infty} (-1)^n u^{1,1}_n(R \times F) + \sum_{n=0}^{\infty} (-1)^n \left( \frac{\rho}{n+1} \right) - \sum_{n=0}^{\infty} (-1)^n u_n(R)
\]

\[
= \sum_{n=0}^{\infty} (-1)^n u^{1,1}_{n+1}(R \times F) + \sum_{n=0}^{\infty} (-1)^n u^{1,1}_n(R \times F) + 1 - \chi(R)
\]

\[
= u^{1,1}_0(R \times F) + 1 - \chi(R)
\]

\[
= 2 - \chi(R)
\]

**Corollary 6.3.** Let \( F_1, \ldots, F_n \) be fields. Then

\[
\chi(F_1 \times \ldots \times F_n) = 1 + (-1)^n
\]
We have not yet found a general method for computing \( \chi(\mathbb{Z}/n\mathbb{Z}) \), where \( n > 0 \) is an arbitrary integer. However, it is possible to analyze some specific examples using idiosyncratic counting methods:

**Theorem 6.4.** Let \( p, q \) be primes and \( r \geq 2 \) an integer. Then

\[
\chi(\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/q^2\mathbb{Z}) = 2 - \chi(\mathbb{Z}/p^r\mathbb{Z}) + \sum_{k=1}^{r-1} \chi(\mathbb{Z}/p^k\mathbb{Z})
\]

**Proof.**

For convenience, set \( R = \mathbb{Z}/p^r\mathbb{Z} \) and \( S = \mathbb{Z}/q^2\mathbb{Z} \); to ease notation, we denote the unique proper ideal of \( S \) by \((q)\). As in Theorem 6.2, let \( \pi_1, \pi_2 \) be the projection maps onto the respective factors of \( R \times S \). As before, for any \( n \geq 0 \), the typical element \( U_n(R \times S) \) is an unordered \((n + 1)\)-tuple \( \{I_0, \ldots, I_n\} \) where \( I_0 \ldots I_n \neq 0 \) and \( I_i = A_i \times B_i \), where \( A_i = \pi_1(I_i) \) an ideal of \( R \) and \( B_i = \pi_2(I_i) \) an ideal of \( S \). In this situation, \( B_i \) may either be 0, \((q)\), or \( S \). As before, \( \prod_{i=0}^n A_i \neq 0 \) or \( \prod_{i=0}^n B_i \neq 0 \).

\[
U_n^1(R \times S) = \{\{I_0, \ldots, I_n\} \in U_n(R \times S) : \prod_{i=0}^n A_i \neq 0\}
\]

\[
U_n^2(R \times S) = \{\{I_0, \ldots, I_n\} \in U_n(R \times S) : \prod_{i=0}^n B_i \neq 0\}
\]

\[
= \{\{I_0, \ldots, I_n\} \in U_n : \text{there exists some } i_0 \text{ such that } B_{i_0} = S \text{ or } B_{i_0} = (q) \text{ and } B_i = S \text{ for all } i \neq i_0\}
\]

\[
U_n^3(R \times S) = U_n^1(R \times S) \cap U_n^2(R \times S)
\]

Now define

\[
U_n^{1,0}(R \times S) = \{\{I_0, \ldots, I_n\} \in U_n(R \times S) : I_i \neq R \times 0 \text{ for all } i, \ 0 \leq i \leq n\}
\]

\[
U_n^{1,1}(R \times S) = U_n^1(R \times S) - U_n^{1,0}(R \times S)
\]

\[
U_n^{3,q}(R \times S) = \{\{I_0, \ldots, I_n\} \in U_n^3(R \times S) : \text{there exists } i_0 \text{ such that } B_{i_0} = (q) \text{ and } B_i = S \text{ for all } i \neq i_0\}
\]
\[ U^3_s(R \times S) = U^3_s(R \times S) - U^3_a(R \times S) \]

\[ = \{ \{ I_0, \ldots, I_n \} \in U^3_s(R \times S) : B_i = S \text{ for all } i, \ 0 \leq i \leq n \} \]

It follows immediately from the above definitions that \( u_n(R \times S) = u^1_n(R \times S) + u^2_n(R \times S) - u^3_n(R \times S) \).

The map \( U^1_n(R \times S) \to U^1_{n+1}(R \times S) \) sending \( \{ I_0, \ldots, I_n \} \mapsto \{ I_0, \ldots, I_n, R \times 0 \} \) establishes a bijection, so \( u^1_n(R \times S) = u^1_{n+1}(R \times S) \).

Now let \( \rho \) denote the number of proper ideals in \( R \). Evidently, by the description given above,

\[ u^2_n(R \times S) = \binom{\rho}{n} + \binom{\rho}{n+1}. \]

Finally, it is clear that \( u^3_s(R \times S) = u_n(R) \). Observe that given a typical element \( \{ I_0, \ldots, I_n \} \) of \( U^3_s(R \times S) \), we may assume without loss of generality that \( B_j = S \) for all \( j > 0 \) and that \( B_0 = (p^k) \times (q) \) for some \( k, 1 \leq k \leq r-1 \). (This is the only place in the proof where we use the fact that \( R \) has the form \( \mathbb{Z}/p^r\mathbb{Z} \).) Thus, in order to have \( \prod_{i=0}^n A_i \neq 0 \), we must have \( \{ A_1, \ldots, A_n \} \in U_{n-1}(\mathbb{Z}/p^{r-k}\mathbb{Z}) \). Hence, \( u^3_a(R \times S) = \sum_{k=1}^{r-1} u_{n-1}(\mathbb{Z}/p^{k}\mathbb{Z}) \).

Collecting this information together, we have:

\[ \chi(R \times S) = \sum_{n=0}^{\infty} (-1)^n u_n(R \times S) \]

\[ = \sum_{n=0}^{\infty} (-1)^n (u^1_n(R \times S) + u^2_n(R \times S) - u^3_n(R \times S)) \]

\[ = \sum_{n=0}^{\infty} (-1)^n (u^1_n(R \times S) + u^1_{n+1}(R \times S) + \binom{\rho}{n} + \binom{\rho}{n+1} - u_n(R) - \sum_{k=1}^{r-1} u_{n-1}(\mathbb{Z}/p^k\mathbb{Z})) \]

\[ = \sum_{n=0}^{\infty} (-1)^n (u^1_n(R \times S) + u^1_{n+1}(R \times S)) + \sum_{n=0}^{\infty} (-1)^n (\binom{\rho}{n} + \binom{\rho}{n+1}) \]

\[ - \sum_{n=0}^{\infty} (-1)^n u_n(R) - \sum_{n=1}^{\infty} (-1)^n \sum_{k=1}^{r-1} u_{n-1}(\mathbb{Z}/p^k\mathbb{Z})) \]
\[ u_0^{1,1}(R \times S) + 1 - \chi(R) + \sum_{k=1}^{r-1} \sum_{n=1}^{\infty} (-1)^{n-1} u_{n-1}(\mathbb{Z}/p^k\mathbb{Z}) \]

\[ = 2 - \chi(R) + \sum_{k=1}^{r-1} \chi(\mathbb{Z}/p^k\mathbb{Z}) \]

Thus

\[ \chi(\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/q^2\mathbb{Z}) = 2 - \chi(\mathbb{Z}/p^r\mathbb{Z}) + \sum_{k=1}^{r-1} \chi(\mathbb{Z}/p^k\mathbb{Z}) \]

From Theorem 6.4 and Theorem 6.1, we see that the value of \( \chi(\mathbb{Z}/p^r\mathbb{Z}) \) may be made arbitrary large by choosing \( r \) large enough. By Theorem 6.2, we see that by taking the product with a field, we can make obtain a ring whose Euler characteristic is arbitrary large and negative. Summarizing, we have:

**Corollary 6.5.** The value of \( \chi(R) \) is unbounded in both the positive and negative directions as \( R \) ranges over the set of finite rings.

It is not difficult to develop *ad hoc* counting methods along similar lines to compute \( \chi(\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/q^s\mathbb{Z}) \), but it is not clear how to generalize this method to compute \( \chi(\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/q^s\mathbb{Z}) \) for arbitrary \( s \geq 1 \).
References


Reza Akhtar
Department of Mathematics and Statistics
Miami University
Oxford, OH 45056
reza@calico.mth.muohio.edu

Lucas Lee
Department of Computer Science and Engineering
University of California, San Diego
San Diego, CA 92093
lalee@ucsd.edu