Homology of zero-divisors

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1 Introduction

Let $R$ be a commutative ring with unity. The set $Z(R)$ of zero-divisors in a ring does not possess any obvious algebraic structure; consequently, the study of this set has often involved techniques and ideas from outside algebra. Several recent attempts, among them [2], [3] have focused on studying the so-called zero-divisor graph $\Gamma_R$, whose vertices are the zero-divisors of $R$, with $xy$ being an edge if and only if $xy = 0$. This object $\Gamma_R$ is somewhat unwieldy in that it has many symmetries; for example, if $u \in R^*$ is any unit, then $x \mapsto ux$ induces a (graph) automorphism of $\Gamma_R$. One way of treating this issue – following an idea of Lauve [5] – is to work with the ideal zero-divisor graph $\mathcal{I}_R$. In effect, one replaces zero-divisors of $R$ by proper ideals with nonzero annihilator; this is the approach adopted by the authors in [1]. Such a perspective also has its shortcomings; for instance, it does not adequately detect the phenomenon of there being three distinct proper ideals $I, J, K$ in $R$ with $IJK = 0$, but $IJ \neq 0$, $IK \neq 0$, $JK \neq 0$.

In this paper we adopt a different philosophy, using a new type of homology to study $Z(R)$ and capture the situation described above. Roughly speaking, if we denote by $\mathbb{Z}_n(R)$ the free abelian group generated by the set of $(n+1)$-tuples $(I_0, \ldots, I_n)$ of distinct ideals of $R$ such that $I_0 \cdots I_n \neq 0$, there are obvious maps $\mathbb{Z}_n(R) \rightarrow \mathbb{Z}_{n-1}(R)$ obtained by forgetting one of the factors. This gives $\mathbb{Z}(R)$ has the structure of a semi-simplicial abelian group; hence we may speak of its associated chain complex $\mathcal{C}(R)$. Our homology groups $H_\ast(R)$ are then defined as the homology groups of a certain quotient of $\mathcal{C}(R)$. The idea behind this construction was sketched by Lauve in [5], although the precise definition is due to the authors.

After giving a precise definition of these homology groups $H_\ast(R)$, we study the group $H_0(R)$ in depth and compute $H_1(\mathbb{Z}/p^r\mathbb{Z})$ when $p$ is a prime and $r \geq 1$ is an integer. We then give some conditions on $R$ sufficient to ensure that $H_n(R) = 0$ for $n > 0$. In the last section we consider the Euler characteristic $\chi(R) = \sum_{n=0}^{\infty} (-1)^n \text{rk } H_n(R)$. Using some ideas from partition theory, we prove the surprising result that $\chi(\mathbb{Z}/p^r\mathbb{Z})$
is always either 0, 1, or 2, depending on the value of \( r \) relative to the “pentagonal” numbers \( m(3m-1)/2 \) and the related numbers \( m(3m+1)/2 \). We also derive formulas for the Euler characteristic for some other special types of finite rings.

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## 2 Preliminaries

Let \( R \) be a commutative ring and \( \mathcal{P} \) the set of proper ideals of \( R \). For each \( n \geq 0 \), let \( S_n(R) \) be the set of ordered \((n+1)\)-tuples \((I_0, \ldots, I_n)\), where \( I_0, \ldots, I_n \) are distinct proper ideals of \( R \) and \( I_0 I_1 \ldots I_n \neq 0 \); let \( S_{-1}(R) \) be a singleton set. If there is no danger of ambiguity, we simply write \( S_n \) instead of \( S_n(R) \). Observe that for each \( i \), \( 0 \leq i \leq n \), there is a “face map” \( \phi^n_i : S_n \to S_{n-1} \) defined by \( \phi^n_i(I_0, \ldots, I_n) = (I_0, \ldots, \hat{I}_i, \ldots, I_n) \) and, if \( S_0 \neq \emptyset \), a unique map \( \varepsilon : S_0 \to S_{-1} \). Note that \( S_0(R) = \emptyset \) if and only if \( R \) is a field. Now for each \( n \geq -1 \), let \( Z_n \) be the free abelian group generated by \( S_n \). We denote by \([I_0, \ldots, I_n]\) the basis element corresponding to \((I_0, \ldots, I_n) \in S_n \). Likewise, the various face maps \( \phi^n_i \) extend \( \mathbb{Z} \)-linearly to maps \( \phi^n_i : Z_n \to Z_{n-1} \); moreover, if \( S_0 \neq \emptyset \), there is a unique \( \mathbb{Z} \)-linear map \( \varepsilon : Z_0 \to Z_{-1} = \mathbb{Z} \) defined by \( \varepsilon(\sum n_i(I_i)) = \sum n_i \). Thus, there is a semisimplicial abelian group:

\[
\mathbf{Z}(R) : \ldots \xrightarrow{\delta_1} Z_1 \xrightarrow{\delta_0} Z_0
\]

with augmentation \( \varepsilon : Z_0 \to \mathbb{Z} \).

This in turn gives rise to an (augmented) chain complex in the standard manner by taking an alternating sum of face maps. For each \( n \geq 0 \), define \( \delta_n = \sum_{i=0}^{n} (-1)^i \phi^n_i \); then we have a complex:

\[
\mathbf{C}(R) : \ldots \xrightarrow{\delta_1} Z_1 \xrightarrow{\delta_0} Z_0
\]

of abelian groups.

In practice, the \( Z_n \) are too large to be useful invariants; in particular, we chose \( Z_n \) to be the free \( \mathbb{Z} \)-module with basis \( S_n \), which consisted of ordered \((n+1)\)-tuples of ideals of \( R \) having nonzero product. Because multiplication in \( R \) is commutative, the order of the ideals in this \((n+1)\)-tuple ought not to matter; it might appear more
natural to work with unordered \((n + 1)\)-tuples. Unfortunately, the definition of the face maps does depend on the ordering within each such tuple, so we resort instead to the following device: for each \(n \geq 0\), let \(R_n\) denote the subgroup of \(Z_n\) generated elements of the form:

\[ [I_0, \ldots, I_n] - (-1)^{\text{sgn } \sigma} [I_{\sigma(0)}, \ldots, I_{\sigma(n)}] \]

where \(\sigma\) in an element of the symmetric group \(\mathfrak{S}_{n+1}\) (viewed as permutations of the set \(\{0, \ldots, n\}\)) and \([I_0, \ldots, I_n]\) is a basis element of \(Z_n\). Set \(T_n = Z_n/R_n\).

For each \(i\), let \(\tau_i \in \mathfrak{S}_n\) be the permutation of \(\{0, \ldots, \hat{\sigma}(i), \ldots, n\}\) defined for \(0 \leq j \leq n\) by:

\[ \tau_i(j) = \sigma(j + \varepsilon_j) - \eta_j \]

where

\[ \varepsilon_j = \begin{cases} 
0 & \text{if } j < i \\
1 & \text{if } j \geq i 
\end{cases} \]

and

\[ \eta_j = \begin{cases} 
0 & \text{if } \sigma(j + \varepsilon_j) < \sigma(i) \\
1 & \text{if } \sigma(j + \varepsilon_j) \geq \sigma(i) 
\end{cases} \]

In effect, \(\tau_i \in \mathfrak{S}_{n-1}\) is the permutation obtained by eliminating \(i\) from the domain of \(\sigma\) and \(\sigma(i)\) from the range of \(\sigma\), and then reindexing appropriately. It is clear from this description that \(\text{sgn } \sigma = \text{sgn } \tau_i\) for all \(i\).

Observe next that

\[ \delta_n((-1)^{\text{sgn } \sigma} [I_{\sigma(0)}, \ldots, I_{\sigma(n)}]) = \sum_{i=0}^{n} (-1)^{\text{sgn } \sigma} [I_{\sigma(0)}, \ldots, \hat{I}_{\sigma(i)}, \ldots, I_{\sigma(n)}] \]

\[ = (-1)^{\text{sgn } \sigma} \sum_{i=0}^{n} (-1)^{\text{sgn } \tau_i} [I_0, \ldots, \hat{I}_i, \ldots, I_n] = \sum_{i=0}^{n} [I_0, \ldots, \hat{I}_i, \ldots, I_n] \]

\[ = \delta_n([I_0, \ldots, I_n]) \]

Thus \(\delta_n(R_n) \subseteq R_{n-1}\) for all \(n \geq 1\), and hence \(\mathcal{C}_*(R)\) factors through a complex:

\[ \mathcal{C}_*(R) : \ldots \overset{\partial_2}{\rightarrow} T_1 \overset{\partial_1}{\rightarrow} T_0 \overset{\varepsilon}{\rightarrow} \mathbb{Z} \rightarrow 0 \]

By abuse of notation, we continue to use the symbol \([I_0, \ldots, I_n]\) to denote the class of \([I_0, \ldots, I_n]\) in \(T_n\).

Finally we define the homology groups:
\[ H_n(R) = \begin{cases} \frac{\text{Ker } \partial_n}{\text{Im } \partial_n} & \text{if } n > 0 \\ \frac{T_n}{\text{Im } \partial_0} & \text{if } n = 0 \end{cases} \]

If \( \text{rk } H_n(R) \) is finite for all \( n \) and zero for sufficiently large \( n \), we define the Euler characteristic of \( R \):

\[ \chi(R) = \sum_{n=0}^{\infty} (-1)^n \text{rk } H_n(R) \]

Since a field has no proper ideals, we immediately have:

**Proposition 2.1.** Let \( F \) be a field. Then \( H_n(F) = 0 \) for all \( n \geq 0 \).

The term “homology” is used somewhat loosely, since neither the complexes \( \mathcal{C}(R) \) nor the groups \( H_n(R) \) are functorial in \( R \). This is not particularly surprising: given a ring homomorphism \( f : R \rightarrow S \), if \( [I_0, \ldots, I_n] \in T_n(R) \), it is possible that \( I_0 \ldots I_n = 0 \) or one of the \( f(I_i) \) may be zero, so it does not necessarily follow that \( [f(I_0), \ldots, f(I_n)] \) makes sense as an element of \( T_n(S) \). Similarly, if \( [J_0, \ldots, J_n] \in T_n(S) \), it does not follow that \( [f^{-1}(I_0), \ldots, f^{-1}(I_n)] \) defines an element of \( T_n(R) \).

The following well-known device is often useful in computing the Euler characteristic:

**Proposition 2.2.** Suppose \( \text{rk } T_n \) is finite for all \( n \) and \( T_n = 0 \) for \( n \gg 0 \). Then

\[ \chi(R) = \sum_{n=0}^{\infty} (-1)^n \text{rk } T_n \]

**Proof.**

By definition of \( H_0(R) \), there is an exact sequence:

\[ 0 \rightarrow \text{Im } \partial_0 \rightarrow T_0 \rightarrow H_0(R) \rightarrow 0 \]

and for each \( n \geq 1 \), there is a short exact sequence:

\[ 0 \rightarrow \text{Im } \partial_n \rightarrow \text{Ker } \partial_{n-1} \rightarrow H_n(R) \rightarrow 0 \]

Since the rank is additive across exact sequences, we have:

\[ \chi(R) = \sum_{n=0}^{\infty} (-1)^n \text{rk } H_n = \text{rk } T_0 - \text{rk } \text{Im } \partial_0 + \sum_{n=1}^{\infty} (-1)^n (\text{rk } \text{Ker } \partial_{n-1} - \text{rk } \text{Im } \partial_n) \]
Furthermore for any \( n \geq 0 \), \( \text{rk Im} \partial_n = \text{rk} T_{n+1} - \text{rk Ker} \partial_n \), so the above expression for \( \chi(R) \) becomes:

\[
\chi(R) = \text{rk} T_0 - \text{rk} T_1 + \text{rk Ker} (\partial_0) + \sum_{n=1}^{\infty} (-1)^n (\text{rk Ker} \partial_{n-1} - \text{rk} T_{n+1} + \text{rk Ker} \partial_n)
\]

\[
= \text{rk} T_0 - \text{rk} T_1 + \sum_{n=1}^{\infty} (-1)^n \text{rk} T_{n+1} = \sum_{n=0}^{\infty} (-1)^n \text{rk} T_n
\]

3 The group \( H_0(R) \)

Let \( R \) be a commutative ring as before. In order to analyze \( H_0(R) \), we recall the construction of the so-called ideal graph \( \mathcal{I}_R \). This is a (simple) graph whose vertices are the proper ideals of \( R \), with \( \{ I, J \} \) being an edge if and only if \( IJ = 0 \). We will be more interested in the complement graph \( \overline{\mathcal{I}}_R \), whose vertices are the same as \( \mathcal{I}_R \), but in which \( \{ I, J \} \) is an edge if and only if \( IJ \neq 0 \).

If \( \sum_{i=1}^{n} [I_i] \in T_0 \) is an element whose class in \( H_0(R) \) is zero, this means that \( \sum_{i=1}^{n} [I_i] = \partial_0(\sum_{j=1}^{m} c_j[A_j, B_j]) \) for some integers \( c_j \) and proper ideals \( A_j, B_j \). Without loss of generality we may assume \( c_j = \pm 1 \). Equality still holds if we replace \( [A_j, B_j] \) by \( -[B_j, A_j] \), so we may always write \( \sum_{i=1}^{n} [I_i] = \partial_0(\sum_{k=1}^{r} [C_k, D_k]) \) for some proper ideals \( C_k, D_k \).

**Proposition 3.1.** Let \( I \) and \( J \) be distinct proper ideals of \( R \). Then \( [I] \) and \( [J] \) have the same class in \( H_0(R) \) if and only if \( I \) and \( J \) lie in the same connected component of the graph \( \overline{\mathcal{I}}_R \).

**Proof.**

If \( I \) and \( J \) are in the same connected component of \( \overline{\mathcal{I}}_R \), then there is some path \( I = A_0 - A_1 - \ldots - A_n = J \) connecting \( I \) and \( J \), where the \( A_i \) are ideals such that for each \( i = 0, \ldots, n-1 \), \( A_iA_{i+1} \neq 0 \). This directly implies that \( \sum_{i=0}^{n-1} [A_i, A_{i+1}] \) is an element of \( T_1 \), and by direct calculation we see that

\[
\partial_0(\sum_{i=0}^{n-1} [A_i, A_{i+1}]) = [A_0] - [A_n] = [I] - [J]
\]

Hence \( [I] = [J] \) in \( H_0(R) \).
Conversely, suppose $[I]$ and $[J]$ define the same class in $H_0(R)$. Then $[I] - [J] = \partial_0(\sum_{i=0}^n [A_i, B_i]) = \sum_{i=0}^n [A_i] - [B_i]$ where $A_i, B_i$ are distinct proper ideals of $R$ and $A_iB_i \neq \emptyset$. Let $n$ be the smallest integer for which this is possible. We prove by induction on $n$ that, after suitable reordering of the $A_i$ and $B_i$, there is a path in $\overline{I}_R$ from $I$ to $J$.

We may assume without loss of generality that $A_0 = I$ and $B_n = J$. If $B_0 = J$, then $IJ \neq 0$ and we are done. Otherwise, assume $B_0 \neq J$; that is, $n > 0$. Since $[I] - [J] = [I] - [B_0] + [A_1] - [B_1] + \ldots + [A_n] - [B_n]$ is a relation in a free abelian group, we may assume without loss of generality that $A_1 = B_0$. Then, adding $[B_0] - [I]$ to both sides of this equation, we get

$$[B_0] - [J] = [A_1] - [B_1] + \ldots + [A_n] - [B_n]$$

so by induction there is a path in $\overline{I}_R$ from $B_0$ to $J$. Since $A_0B_0 \neq 0$, this means that $\{A_0, B_0\}$ is an edge in $\overline{I}_R$, and hence that there is a path from $A_0 = I$ to $J$.

Proposition 3.2. Let $I_1, \ldots, I_n$ be distinct proper ideals of $R$ lying in mutually distinct connected components of $\overline{I}_R$. Then the classes of $[I_1], \ldots, [I_n]$ are linearly independent in $H_0(R)$.

Proof.
If $R$ is a field, the assertion is trivial. Otherwise, let $C_1, \ldots, C_r$ be the components of $\overline{I}_R$. Suppose the class of $\sum_{i=1}^n c_i[I_i]$ in $H_0(R)$ is 0. We may assume that each $I_i$ lies in component $C_i$ of $\overline{I}_R$. Now

$$\sum_{i=1}^n c_i[I_i] = \partial_0(\sum_{j=1}^m [A_j, B_j])$$

for some distinct proper ideals $A_j, B_j$ such that $A_jB_j \neq 0$. Since $[A_j, B_j] \in T_1$, $A_j$ and $B_j$ must lie in the same component of $\overline{I}_R$. For each $k$, $1 \leq k \leq r$, let $J_k = \{j : 1 \leq j \leq m : A_j \in C_k\}$. Then it follows from the above equation that

$$c_k[I_k] = \partial_0(\sum_{j \in J_k} [A_j] - [B_j])$$

Applying $\varepsilon$ to both sides of this equation, we have $c_k = 0$ for all $k$.

Combining the previous two propositions, we have:
Corollary 3.3. Let $R$ be a ring, and $r$ the number of connected components of $\mathcal{I}_R$. Then
\[ H_0(R) \cong \mathbb{Z}^r. \]

Corollary 3.3 is a useful tool for calculating $H_0(R)$ in particular cases; nevertheless, using only elementary facts about ideals, one can prove even more. We begin with an elementary lemma:

Lemma 3.4. Let $R$ be a ring and $m_1, m_2$ distinct maximal ideals of $R$. If $m_1 m_2 = 0$, then $R$ is isomorphic to a product of two fields.

Proof. Let $p$ be a prime ideal of $R$. Then $p \supseteq m_1 m_2 = 0$, so $p \supseteq m_1$ or $p = m_2$. Hence $m_1$ and $m_2$ are the only prime ideals of $R$ and so $R$ is an Artin ring with two maximal ideals. By the structure theorem for Artin rings, $R \cong R_1 \times R_2$, where $R_1, R_2$ are Artin local rings with respective maximal ideals $n_1, n_2$. Then without loss of generality, $m_1 = n_1 \times R_2$ and $m_2 = R_1 \times n_2$. Thus, $0 = m_1 m_2 = n_1 \times n_2$ so $n_1 = 0, n_2 = 0$ and so $R_1, R_2$ are fields.

Proposition 3.5. Let $R$ be a nonlocal ring which is not isomorphic to the product of two fields. Then $H_0(R) \cong \mathbb{Z}$.

Proof. By Corollary 3.3 it suffices to prove that $\mathcal{I}_R$ is connected. Indeed, let $m_1, m_2$ be distinct maximal ideals of $R$. If $I$ is any other proper ideal of $R$, then $\text{ann}(I)$ is a proper ideal of $R$, so $\text{ann}(I)$ does not contain both $m_1$ and $m_2$. Hence for each such $I$, at least one of $\{I, m_1\}, \{I, m_2\}$ is an edge in $\mathcal{I}_R$. If $m_1 m_2 = 0$, then it follows from Lemma 3.4 that $R$ is isomorphic to a product of two fields. Thus $m_1 m_2 \neq 0, \{m_1, m_2\}$ is an edge of $\mathcal{I}_R$, and it follows that $\mathcal{I}_R$ is connected.

We have seen that $H_0(F) = 0$ when $F$ is a field and $H_0(R) \cong \mathbb{Z}$ for a large class of rings. Direct computation shows that if $F_1$ and $F_2$ are fields, then $H_0(F_1 \times F_2) \cong \mathbb{Z}^2$ and $H_n(F_1 \times F_2) = 0$ for all $n > 0$. A natural question that arises is: given any integer $s \geq 0$, is there a ring $R$ such that $H_0(R) \cong \mathbb{Z}^s$? The discussion above shows that when $s \geq 3$, any such $R$ must necessarily be local. Following an idea supplied to us by Dennis Keeler, we show below that the rank of $H_0(R)$ may be arbitrarily large.

Let $k$ be a field and $x_1, \ldots, x_s$ independent indeterminates. Let $S$ be the localization of $k[x_1, \ldots, x_s]$ with respect to the maximal ideal $(x_1, \ldots, x_s)$. Now let $I$ be the ideal of $k[x_1, \ldots, x_s]$ generated by all products $x_i x_j$, where $i \leq j$. Since $I \subseteq (x_1, \ldots, x_s)$, $I$ corresponds, in the usual manner, to an ideal $\tilde{I} \subseteq S$. Now let $R = S/\tilde{I}$. Observe now that
the proper ideals of $R$ correspond bijectively to ideals $(x_{i_1}, \ldots, x_{i_\nu}) \subseteq k[x_1, \ldots, x_s]$, where $1 \leq \nu \leq s$ and $1 \leq i_1 < \ldots < i_\nu \leq s$. Furthermore, each such ideal (of $R$), when multiplied by any other, yields 0. Thus $\overline{I}_R$ is a completely disconnected graph on $2^s - 2$ vertices, and so $H_0(R) \cong \mathbb{Z}^{2^s - 2}$.

4 Calculation of $H_1(\mathbb{Z}/p^r\mathbb{Z})$

In this section, we compute the group $H_1(\mathbb{Z}/p^r\mathbb{Z})$ where $p$ is a prime number and $r \geq 1$ an integer. It is easy to see by direct calculation that if $r \leq 3$, then $H_1(\mathbb{Z}/p^r\mathbb{Z}) = 0$. We assume henceforth that $r \geq 4$.

Recall first that

$$H_1(R) = \frac{\ker (\partial_0 : T_1 \to T_0)}{\im (\partial_1 : T_2 \to T_1)}$$

where

$$\partial_0(\sum_j [A_j, B_j]) = \sum_j [A_j] - [B_j]$$

and

$$\partial_1(\sum_j [A_j, B_j, C_j]) = \sum_j [B_j, C_j] - \sum_j [A_j, C_j] + \sum_j [A_j, B_j]$$

**Definition 4.1.** Let $n \geq 0$ be an integer. An element $\alpha \in T_1$ is called an $n$-circuit (or simply a circuit) if there exist proper ideals $I_1, \ldots, I_n$ of $R$ such that

$$\alpha = [I_1, I_2] + \ldots + [I_{n-1}, I_n] + [I_n, I_1]$$

A 3-circuit is called a triangle.

Clearly the definition has been chosen to reflect the fact that in the above context, $I_1 - I_2 - \ldots I_n - I_1$ is a circuit in the graph $\overline{I}_{\mathbb{Z}/p^r\mathbb{Z}}$. The analysis of $\ker \partial_0$ proceeds by a sequence of lemmas.

**Lemma 4.2.** Every element $\beta \in \ker \partial_0$ may be written

$$\beta = \sum_{k=1}^m \alpha_k$$

where each $\alpha_k$ is a circuit.
Proof.
The proof is by induction on the number of symbols in $\beta$. If $\beta = 0$, the claim is clear. Otherwise, let $\beta = \sum_{j=1}^{r} [A_j, B_j]$ with $r$ chosen to be as small as possible. We may assume that there is no pair of integers $(j_1, j_2)$, $1 \leq j_1 < j_2 \leq r$ such that $A_{j_1} = B_{j_2}$ and $A_{j_2} = B_{j_1}$, for then we may use the relation $[I, J] = -[J, I]$ in $T_1$ to simplify the expression for $\beta$ and obtain a relation with smaller $r$.

Since $\beta \in \ker \partial_0$, we have:

$$0 = \partial_0(\beta) = \partial_0(\sum_{j=1}^{r} [A_j, B_j]) = \sum_{j=1}^{r} [A_j] - [B_j]$$

Since this is a relation in the (free abelian) group $T_0$, it follows that there is some $j$ such that $B_1 = A_j$. Without loss of generality we may assume that $j = 2$. By the previous discussion, it follows that $A_1 \neq B_2$. Now it must be the case that there is some $j$ such that $B_2 = A_j$; without loss of generality, we assume that $j = 3$. Continue this procedure until one reaches $s \leq r$ such that $B_s = A_1$. Then

$$\beta_1 = [A_1, B_1] + [B_1, B_2] + \ldots + [B_{s-2}, B_{s-1}] + [B_{s-1}, A_1]$$

is a circuit in $T_1$. By induction, $\beta - \beta_1$ is a sum of circuits in $T_1$; hence $\beta$ itself is a sum of circuits.

Lemma 4.3. Every nonzero circuit in $T_1 = T_1(\mathbb{Z}/p^r\mathbb{Z})$ may be written as a sum of triangles.

Proof.
Let $\alpha = \sum_{j=1}^{r-1} [A_j, A_{j+1}] + [A_r, A_1]$ be a circuit in $T_1$. If $\alpha$ is a 3-circuit, there is nothing to prove. By induction, it suffices to prove that $\alpha$ has a chord, i.e. there exist distinct integers $i$, $j$, $1 \leq i < j \leq r$ such that $[A_i, A_j] \in T_1$ and $j - i > 1$. Suppose $\alpha$ is an $n$-circuit, with $n > 3$. For each $k$, $1 \leq k \leq r - 1$, let $I_k$ denote the ideal of $\mathbb{Z}/p^r\mathbb{Z}$ generated by (the class of) $p^k$. Let $S = \{I_k : 1 \leq k < r/2\}$. Observe that if $C, D \in S$, then $[C, D] \in T_1$. Furthermore, if $[C, D] \in T_1$ and $C \notin S$, then $D$ must be in $S$.

If all the $A_i$ appearing in the cycle $\alpha$ are members of $S$, then by the above observation $[A_1, A_2] + [A_2, A_3] + [A_3, A_1]$ is a triangle. If not, then we may assume without loss of generality that $A_2 \notin S$. Since $[A_1, A_2] \in T_1$ and $[A_2, A_3] \in T_1$, we must have $A_1 \in S$, $A_3 \in S$. This forces $[A_1, A_3] \in T_1$, which completes the proof.
Lemma 4.4. Every triangle in $T_1(\mathbb{Z}/p^r\mathbb{Z})$ may be written as a sum of triangles of the form \( \tau_{ij} = [I_1, I_i] + [I_i, I_j] + [I_j, I_1] \), where \( 1 < i, j < r \).

Proof.
This follows immediately from the formal identity:

\[
[I_h, I_i] + [I_i, I_j] + [I_j, I_h] = (\tau_{hi} + \tau_{ij} + \tau_{jh})
\]

Lemma 4.5. The set of triangles $T = \{ \tau_{ij} : 1 < i < j < r \}$ is ($\mathbb{Z}$)-linearly independent in $T_1$.

Proof.
This follows readily from the fact that $\tau_{ij}$ is the only member of $T$ involving the symbol $[I_i, I_j]$.

It follows from the sequence of lemmas above that:

Corollary 4.6. The group $\text{Ker} \partial_0$ is a free abelian group with basis $T$.

In fact, $\tau_{ij} \in T$ if and only if $i + j < r$, so an elementary counting argument gives:

Corollary 4.7. The rank of $\text{Ker} \partial_0$ is $\frac{(r - 4)^2}{4}$ if $r$ is even or $\frac{(r - 4)^2 - 1}{4}$ if $r$ is odd.

We now examine the group $\text{Im} \partial_1$. Observe that:

$\gamma = \partial_1([I_i, I_j, I_k]) = [I_i, I_j] - [I_i, I_k] + [I_j, I_k] = [I_i, I_j] + [I_j, I_k] + [I_k, I_i]$

is a triangle of $T_1$.

Since $I_i I_j I_k \neq 0$ and $I_1$ contains $I_i$, $I_j$ and $I_k$, it follows readily that each of the symbols $[I_1, I_i, I_j]$, $[I_1, I_i, I_k]$ and $[I_1, I_j, I_k]$ are in $T_2$; furthermore,

\[
\gamma = \partial_1([I_i, I_j, I_k]) = \partial_1([I_i, I_j]) + \partial_1([I_j, I_k]) + \partial_1([I_1, I_k, I_i]) = \tau_{ij} + \tau_{jk} + \tau_{ki}
\]

so in fact $\text{Im} \partial_1$ is generated by those elements $\tau_{ij} \in T$ such that $1 + i + j < r$, i.e. $i + j < r - 1$.

By the same computation as used to derive Corollary 4.7, we obtain:
Corollary 4.8. The group $\text{Im} \partial_1$ is a free abelian group of rank 
\[
\frac{(r - 5)^2 - 1}{4}
\] if $r$ is even or 
\[
\frac{(r - 5)^2}{4}
\] if $r$ is odd.

In particular, we observe that the basis elements $\tau_{ij}$ for $\text{Im} (\partial_1)$ identified in the previous discussion are a subset of those identified as a basis for $\text{Ker} (\partial_0)$. Thus, we have:

Corollary 4.9. Suppose $r \geq 4$. Then $H_1(\mathbb{Z}/p^r\mathbb{Z})$ is a free abelian group of rank 
\[
\frac{r - 4}{2}
\] if $r$ is even or 
\[
\frac{r - 5}{2}
\] if $r$ is odd.

5 Acyclicity

In this section, we make a general study of the higher homology groups $H_n(R)$, $n > 0$; in particular, we give various conditions sufficient for these groups to be zero.

Towards this end, it is convenient to introduce some notation: if $I_{j_0}, \ldots, I_{j_m}$ ($j = 1 \ldots r$) and $J_0, \ldots, J_n$ are mutually distinct ideals of a ring $R$ such that $[I_{j_0}, \ldots, I_{j_m}] \in T_m(R)$ for each $j$ and $[J_0, \ldots, J_n] \in T_n(R)$, and also $I_{j_0} \ldots I_{j_m} J_0 \ldots J_n \neq 0$, for each $j$, we write:

\[
\sum_{j=1}^{r} [I_{j_0}, \ldots, I_{j_m}] \times [J_0, \ldots, J_n] = \sum_{j=1}^{r} [I_{j_0}, \ldots, I_{j_m}, J_0, \ldots, J_n]
\]

Lemma 5.1. (Acyclicity Lemma) Suppose $n > 0$ and $\alpha = \sum_{j=1}^{r} [I_{j_0}, \ldots, I_{j_m}] \in \text{Ker} (\partial_{n-1})$. If there exists an ideal $J \notin \{I_{j_k} : 1 \leq j \leq r, 0 \leq k \leq n\}$ such that $JI_{j_0} \ldots I_{j_m} \neq 0$ for all $j$, $1 \leq j \leq r$, then $\alpha \in \text{Im} (\partial_n)$. Thus the class of $\alpha$ in $H_n(R)$ is zero.

Proof.

If such $J$ exists, then

\[
\partial_n((-1)^{n+1} \sum_{j=1}^{r} [I_{j_0}, \ldots, I_{j_m}] \times [J]) = (-1)^{n+1} \sum_{i=0}^{n} (-1)^i [I_{j_0}, \ldots, I_{j_i}, \hat{I}_j, \ldots, I_{j_m}, J] + \alpha
\]

\[
= -\partial_{n-1}(\alpha) \times [J] + \alpha = \alpha
\]

So indeed $\alpha \in \text{Im} (\partial_n)$, as desired.

Theorem 5.2. Let $R$ be a ring satisfying at least one of the following conditions:
There exists a nonzero element $x \in R$ which is neither a unit nor a zero-divisor.

$R$ has infinitely many maximal ideals.

$R$ is reduced, Noetherian, and of positive (Krull) dimension.

Then $H_n(R) = 0$ for all $n > 0$.

Proof.
First, suppose $x \in R$ is a nonzero element which is neither a unit nor a zero-divisor. Then it is easy to see that $x^i$ and $x^j$ are associate if and only if $i = j$. Thus, $(x^i) \supset (x^2) \supset (x^3) \supset \ldots$

is a descending chain of distinct ideals. Furthermore, if $I$ is a nonzero ideal, then $(x^i)I \neq 0$, for any $i \geq 1$ because $x$ (and hence $x^i$) is not a zero-divisor. Given any $n > 0$ and $\alpha = \sum_{j=1}^{r} [I_{j_0}, \ldots, I_{j_n}] \in \text{Ker } (\partial_{n-1})$ as in Lemma 5.1, choose $m$ such that $(x^m) \neq I_{jk}$ for all $j, k$. Then $J = (x^m)$ satisfies the hypotheses of the Lemma and the assertion follows.

Now suppose $R$ has infinitely many maximal ideals, and suppose $\alpha$ is as above. For each $j$, let $A_j = \text{ann}(I_{j_0} \ldots I_{j_n})$; $A_j$ is a proper ideal of $R$, so choose some maximal ideal $m_j$ such that $A_j \subseteq m_j$. For each $j, 1 \leq j \leq r$ and $k, 1 \leq k \leq n$, choose a maximal ideal $m_{jk}$ such that $I_{jk} \subseteq m_{jk}$. Now let

$$D = \bigcup_{j=1}^{r} m_j \cup \bigcup_{j=1}^{r} \bigcup_{k=1}^{n} m_{jk}$$

Let $m$ be some other maximal ideal of $R$ not equal to any $m_j$ or $m_{jk}$. By [4], Proposition 1.11, $m \nsubseteq D$. Choose $x \in m - D$. Evidently, $(x)$ is a proper ideal of $R$. Furthermore, since $x \notin m_{jk}$, $(x) \neq I_{jk}$ for any $j, k$. Finally, $x \notin m_j \supseteq A_j$ implies that $(x)I_{j_0} \ldots I_{j_n} \neq 0$ for all $j$. Thus, $J = (x)$ satisfies the hypotheses of Lemma 5.1, and the assertion is proved.

Last, suppose $R$ is reduced, Noetherian, and $\dim R > 0$. Let $p_0$ be a minimal prime ideal of $R$ which is not also maximal. Then $\dim(R/p_0) > 0$, so in particular $R/p_0$ is not Artinian. Thus, there is a strictly descending sequence of ideals of $R$:

$$R \supseteq J_1 \supseteq J_2 \supseteq \ldots$$

each of which strictly contains $p_0$. 

12
Let \( p_0, \ldots, p_n \) be the minimal prime ideals of \( R \); there are only finitely many of them because \( R \) is Noetherian ([4], Chapter 6, Exercise 9). It is well-known (cf. [4], Prop. 1.8) that the nilradical of \( R \) is the intersection of the prime ideals of \( R \) – hence also of the minimal prime ideals of \( R \). Thus in our case, \( \bigcap_{i=0}^{n} p_i = 0. \)

**Lemma 5.3.** Let \( I \) be any nonzero ideal of \( R \) and \( m \geq 1 \) an integer. Then \( IJ_m \neq 0. \)

**Proof.**
Suppose \( IJ_m = 0. \) Since \( \bigcap_{i=0}^{n} p_i = 0, \) this means \( p_i \supseteq IJ_m \) for each \( i. \) Since \( p_i \) is prime, \( p_i \supseteq I \) or \( p_i \supseteq J_m. \) In the latter case, \( p_i \supseteq J_m \supseteq p_0, \) so by minimality of \( p_i, \) we must have \( p_i = J_m = p_0. \) However, \( J_m \) strictly contains \( p_0, \) so this is impossible. Thus, we must have \( p_i \supseteq I \) for each \( i; \) hence, \( 0 = \bigcap_{i=0}^{n} p_i \supseteq I \) and so \( I = 0. \)

Returning to the proof of Theorem 5.2, suppose \( n > 0 \) and \( \alpha = \sum_{j=1}^{r} [I_{j_0}, \ldots, I_{j_n}] \in \text{Ker} (\partial_{n-1}) \) as in Lemma 5.1. Choose \( m \geq 1 \) such that \( J_m \notin \{ I_{j_k} : 1 \leq j \leq r, 0 \leq k \leq n \}. \) Then for any \( j, 1 \leq j \leq r, JI_{j_0} \ldots I_{j_n} \neq 0 \) by Lemma 5.3, so we may take \( J = J_m \) and apply Lemma 5.1 to conclude.

### 6 \( \chi \) for finite rings

Theorem 5.2 establishes that the higher homology groups are uninteresting for a large class of rings. Finite rings, on the other hand, satisfy none of the conditions of the theorem; in this section, we examine these rings more closely. While the prospect of computing the actual homology groups seems daunting, the Euler characteristic turns out to be a much more tractable object. In particular, if \( R \) is a finite ring – hence having only finitely many ideals – it is clear from the definition that each \( T_n(R) \) has finite rank and that \( T_n(R) = 0 \) for sufficiently large \( n. \) Hence the hypotheses of Proposition 2.2 are satisfied and we may use it to compute the Euler characteristic. In particular, let \( U_n = U_n(R) \) denote the number of unordered \((n+1)\)-tuples \( \{I_0, \ldots, I_n\} \) of distinct ideals whose product is nonzero. Then we have the convenient formula

\[
\chi(R) = \sum_{n=0}^{\infty} (-1)^n |U_n|
\]

Throughout this section, if a set is denoted by an uppercase letter, we will use the corresponding lower case letter for the number of elements in that set. For example, we will write \( u_n \) for \( |U_n| \) as defined above.
We begin by examining the same rings encountered in Sec. 4, namely those of the form \( R = \mathbb{Z}/p^r\mathbb{Z} \) where \( p \) is a prime and \( r \geq 1 \) is some integer. Recall that for each \( i, 1 \leq i \leq r - 1 \), there is an ideal \( I_i \) of \( R \) generated by (the class of) \( (p^i) \) and that these are all the proper ideals of \( R \). In the following, we implicitly identify the ideal \( I_i \) with the integer \( i \). Since \( U_n \) is the set of unordered \((n + 1)\)-tuples \( \{I_0, \ldots, I_n\} \) of distinct proper ideals of \( R \), we have

\[
u_n = \sum_{k=1}^{r-1} P(k, n+1)
\]

where \( P(k, n+1) \) represents the number of partitions of \( k \) into \((n+1)\) distinct positive integer parts. Hence

\[
\chi(R) = \sum_{n=0}^{\infty} (-1)^n s_n = \sum_{n=0}^{\infty} (-1)^n \sum_{k=1}^{r-1} P(k, n+1) = \sum_{k=1}^{r-1} \sum_{n=1}^{\infty} (-1)^{n+1} P(k, n)
\]

We may interpret the inner sum \( \sum_{n=1}^{\infty} (-1)^{n+1} P(k, n) = -\sum_{n=1}^{\infty} (-1)^n P(k, n) \) as the coefficient of \( x^k \) in the power series:

\[-(1-x)(1-x^2)(1-x^3)\ldots\]

By Euler’s pentagonal theorem, we have:

\[-(1-x)(1-x^2)(1-x^3)\ldots = -1 + x + x^2 - x^5 - x^7 + x^{12} + x^{15} - x^{22} - x^{26} + \ldots\]

where the pattern of signs on the right (from the second term forth) is \(+ + - -\) and the exponents alternate between the “pentagonal” numbers of the form \( P_m = \frac{m(3m-1)}{2} \) and the related numbers \( Q_m = \frac{m(3m+1)}{2} \), where \( m = 1, 2, 3, \ldots \).

Hence

\[\chi(R) = \sum_{k=1}^{r-1} -\sum_{n=1}^{\infty} (-1)^n P(k, n)\]

is the sum of the coefficients of the terms \( x, x^2, \ldots, x^{r-1} \) appearing in the above series.

It is clear from the sign pattern that this sum is either 0, 1, or 2, depending on the value of \( r \) in relation to the numbers \( P_m \) and \( Q_m \).

We summarize our findings in the following:
Theorem 6.1. Let $p$ be a prime and $r \geq 1$ an integer. Then $\chi(\mathbb{Z}/p^r\mathbb{Z})$ is equal to 0, 1, or 2, depending on the value of $r$ in relation to the various pentagonal numbers $m(3m-1)/2$ and the associated numbers $m(3m+1)/2$.

By being careful with counting methods, we can prove the following theorem, whose proof is facilitated by the paucity of ideals in a field.

Theorem 6.2. Let $R$ be a finite ring and $F$ a field. Then

$$\chi(R \times F) = 2 - \chi(R)$$

Proof.

Let $\pi_1$, $\pi_2$ denote the projection maps onto the respective factors of $R \times F$. Recall that for any $n \geq 0$, the typical element $U_n(R \times F)$ is an unordered $(n + 1)$-tuple $\{I_0, \ldots, I_n\}$ where $I_0 \ldots I_n \neq 0$. Moreover, each $I_i = A_i \times B_i$, with $A_i = \pi_1(I_i)$ being an ideal of $R$ and $B_i = \pi_2(I_i)$ an ideal of $F$, i.e. $B_0 = 0$ or $B_3 = F$. In order to have $I_0 \ldots I_n \neq 0$, at least one of $\prod_{i=0}^{n} A_i \neq 0$ or $\prod_{i=0}^{n} B_i \neq 0$. Define:

$$U^1_n(R \times F) = \{\{I_0, \ldots, I_n\} \in U_n(R \times F) : \prod_{i=0}^{n} A_i \neq 0\}$$

$$U^2_n(R \times F) = \{\{I_0, \ldots, I_n\} \in U_n(R \times F) : \prod_{i=0}^{n} B_i \neq 0\} = \{\{I_0, \ldots, I_n\} \in U_n : B_i = F \text{ for each } i\}$$

$$U^3_n(R \times F) = U^1_n(R \times F) \cap U^2_n(R \times F)$$

$$= \{\{I_0, \ldots, I_n\} \in U_n(R \times F) : B_i = F \text{ for each } i \text{ and } (A_0, \ldots, A_n) \in U_n(R)\}$$

Thus we have $u_n = u^1_n + u^2_n - u^3_n$.

It is clear from the above description that $u^3_n(R \times F) = u_n(R)$ and furthermore that if $\{I_0, \ldots, I_n\} \in U^2_n(R \times F)$, then $A_0, \ldots, A_n$ are allowed to be any (mutually distinct) proper ideals of $R$; hence $u^2_n(R \times F) = \binom{\rho}{n+1}$, where $\rho$ is the number of proper ideals in $R$.

The set $U^1_n$ is slightly more difficult to analyze: define

$$U^{1,0}_n(R \times F) = \{\{I_0, \ldots, I_n\} \in U^1_n(R \times F) : I_i \neq R \times 0 \text{ for all } i, \ 0 \leq i \leq n\}$$
Clearly \( u_n^{1,0}(R \times F) + u_n^{1,1}(R \times F) = u_n^1(R \times F) \). Somewhat more subtly, there is a natural bijective map \( U_n^{1,0}(R \times F) \to U_{n+1}^{1,1}(R \times F) \) sending \( \{I_0, \ldots, I_n, R \times 0\} \mapsto \{I_0, \ldots, I_n, R \times 0\} \), so it is also true that \( u_n^{1,0}(R \times F) = u_{n+1}^{1,1}(R \times F) \).

Combining all these relations, we have:

\[
\chi(R \times F) = \sum_{n=0}^{\infty} (-1)^n u_n(R \times F)
\]

\[
= \sum_{n=0}^{\infty} (-1)^n (u_n^1(R \times F) + u_n^2(R \times F) - u_n^3(R \times F))
\]

\[
= \sum_{n=0}^{\infty} (-1)^n (u_n^{1,0}(R \times F) + u_n^{1,1}(R \times F) + \binom{\rho}{n+1} - u_n(R))
\]

\[
= \sum_{n=0}^{\infty} (-1)^n u_n^{1,0}(R \times F) + \sum_{n=0}^{\infty} (-1)^n u_n^{1,1}(R \times F) + \sum_{n=0}^{\infty} (-1)^n \binom{\rho}{n+1} - \sum_{n=0}^{\infty} (-1)^n u_n(R)
\]

\[
= \sum_{n=0}^{\infty} (-1)^n u_n^{1,1}(R \times F) + \sum_{n=0}^{\infty} (-1)^n u_n^{1,1}(R \times F) + 1 - \chi(R)
\]

\[
= u_0^{1,1}(R \times F) + 1 - \chi(R)
\]

\[
= 2 - \chi(R)
\]

**Corollary 6.3.** Let \( F_1, \ldots, F_n \) be fields. Then

\[
\chi(F_1 \times \ldots F_n) = 1 + (-1)^n
\]

We have not yet found a general method for computing \( \chi(\mathbb{Z}/n\mathbb{Z}) \), where \( n > 0 \) is an arbitrary integer. However, it is possible to analyze some specific examples using idiosyncratic counting methods:

**Theorem 6.4.** Let \( p, q \) be primes and \( r \geq 2 \) an integer. Then

\[
\chi(\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/q^2\mathbb{Z}) = 2 - \chi(\mathbb{Z}/p^r\mathbb{Z}) + \sum_{k=1}^{r-1} \chi(\mathbb{Z}/p^k\mathbb{Z})
\]
Proof.

For convenience, set $R = \mathbb{Z}/p\mathbb{Z}$ and $S = \mathbb{Z}/q^2\mathbb{Z}$; to ease notation, we denote the unique proper ideal of $S$ by $(q)$. As in Theorem 6.2, let $\pi_1, \pi_2$ be the projection maps onto the respective factors of $R \times S$. As before, for any $n \geq 0$, the typical element $U_n(R \times S)$ is an unordered $(n + 1)$-tuple $\{I_0, \ldots, I_n\}$ where $I_0 \ldots I_n \neq 0$ and $I_i = A_i \times B_i$, where $A_i = \pi_1(I_i)$ an ideal of $R$ and $B_i = \pi_2(I_i)$ an ideal of $S$. In this situation, $B_i$ may either be 0, $(q)$, or $S$. As before, $\prod_{i=0}^{n} A_i \neq 0$ or $\prod_{i=0}^{n} B_i \neq 0$.

$U_1^n(R \times S) = \{\{I_0, \ldots, I_n\} \in U_n(R \times S) : \prod_{i=0}^{n} A_i \neq 0\}$

$U_2^n(R \times S) = \{\{I_0, \ldots, I_n\} \in U_n(R \times S) : \prod_{i=0}^{n} B_i \neq 0\}$

$= \{\{I_0, \ldots, I_n\} \in U_n : \text{there exists some } i_0 \text{ such that } B_{i_0} = S \text{ or } B_{i_0} = (q) \text{ and } B_i = S \text{ for all } i \neq i_0\}$

$U_3^n(R \times S) = U_1^n(R \times S) \cap U_2^n(R \times S)$

Now define

$U_{1,0}^n(R \times S) = \{\{I_0, \ldots, I_n\} \in U_1^n(R \times S) : I_i \neq R \times 0 \text{ for all } i, \ 0 \leq i \leq n\}$

$U_{1,1}^n(R \times S) = U_1^n(R \times S) - U_{1,0}^n(R \times S)$

$U_{3,q}^n(R \times S) = \{\{I_0, \ldots, I_n\} \in U_3^n(R \times S) : \text{there exists } i_0 \text{ such that } B_{i_0} = (q) \text{ and } B_i = S \text{ for all } i \neq i_0\}$

$U_{3,S}^n(R \times S) = U_3^n(R \times S) - U_{3,q}^n(R \times S)$

$= \{\{I_0, \ldots, I_n\} \in U_3^n(R \times S) : B_i = S \text{ for all } i, \ 0 \leq i \leq n\}$

It follows immediately from the above definitions that $u_n(R \times S) = u_{1,1}^n(R \times S) + u_{3,S}^n(R \times S) - u_{3,q}^n(R \times S)$. 

17
The map \( U_{n}^{1,0}(R \times S) \rightarrow U_{n+1}^{1,1}(R \times S) \) sending \( \{I_0, \ldots , I_n\} \mapsto \{I_0, \ldots , I_n, R \times 0\} \) establishes a bijection, so \( u_{n}^{1,0}(R \times S) = u_{n+1}^{1,1}(R \times S) \).

Now let \( \rho \) denote the number of proper ideals in \( R \). Evidently, by the description given above,

\[
u_{n}^{2}(R \times S) = \rho \binom{n}{n} + \binom{\rho}{n+1}.
\]

Finally, it is clear that \( u_{n}^{3,q}(R \times S) = u_{n}(R) \). Observe that given a typical element \( \{I_0, \ldots , I_n\} \) of \( U_{n}^{3,q}(R \times S) \), we may assume without loss of generality that \( B_j = S \) for all \( j > 0 \) and that \( B_0 = (p^k) \times (q) \) for some \( k, 1 \leq k \leq r - 1 \). (This is the only place in the proof where we use the fact that \( R \) has the form \( \mathbb{Z}/p^r \mathbb{Z} \).) Thus, in order to have \( \prod_{i=0}^{n} A_i \neq 0 \), we must have \( \{A_1, \ldots , A_n\} \in U_{n-1}(\mathbb{Z}/p^{r-k} \mathbb{Z}) \). Hence,

\[
u_{n}^{3,q}(R \times S) = \sum_{k=1}^{r-1} u_{n-1}(\mathbb{Z}/p^k \mathbb{Z}).
\]

Collecting this information together, we have:

\[
\chi(R \times S) = \sum_{n=0}^{\infty} (-1)^n u_{n}(R \times S)
\]

\[
= \sum_{n=0}^{\infty} (-1)^n (u_{n}^{1}(R \times S) + u_{n}^{2}(R \times S) - u_{n}^{3}(R \times S))
\]

\[
= \sum_{n=0}^{\infty} (-1)^n (u_{n}^{1,0}(R \times S) + u_{n}^{1,1}(R \times S) + \rho \binom{n}{n} + \binom{\rho}{n+1} - u_{n}(R) - \sum_{k=1}^{r-1} u_{n-1}(\mathbb{Z}/p^k \mathbb{Z}))
\]

\[
= \sum_{n=0}^{\infty} (-1)^n (u_{n}^{1,0}(R \times S) + u_{n}^{1,1}(R \times S)) + \sum_{n=0}^{\infty} (-1)^n \rho \binom{n}{n} + \binom{\rho}{n+1} - \sum_{n=0}^{\infty} u_{n}(R) - \sum_{k=1}^{r-1} \sum_{n=1}^{\infty} u_{n-1}(\mathbb{Z}/p^k \mathbb{Z}))
\]

\[
= u_{0}^{1,1}(R \times S) + 1 - \chi(R) + \sum_{k=1}^{r-1} \sum_{n=1}^{\infty} (-1)^{n-1} u_{n-1}(\mathbb{Z}/p^k \mathbb{Z})
\]

\[
= 2 - \chi(R) + \sum_{k=1}^{r-1} \chi(\mathbb{Z}/p^k \mathbb{Z})
\]

Thus

\[
\chi(\mathbb{Z}/p^r \mathbb{Z} \times \mathbb{Z}/q^2 \mathbb{Z}) = 2 - \chi(\mathbb{Z}/p^r \mathbb{Z}) + \sum_{k=1}^{r-1} \chi(\mathbb{Z}/p^k \mathbb{Z})
\]
From Theorem 6.4 and Theorem 6.1, we see that the value of $\chi(\mathbb{Z}/p^r\mathbb{Z})$ may be made arbitrary large by choosing $r$ large enough. By Theorem 6.2, we see that by taking the product with a field, we can make obtain a ring whose Euler characteristic is arbitrary large and negative. Summarizing, we have:

**Corollary 6.5.** The value of $\chi(R)$ is unbounded in both the positive and negative directions as $R$ ranges over the set of finite rings.

It is not difficult to develop *ad hoc* counting methods along similar lines to compute $\chi(\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/q^s\mathbb{Z})$, but it is not clear how to generalize this method to compute $\chi(\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/q^s\mathbb{Z})$ for arbitrary $s \geq 1$. 
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