(b) Suppose that \( B \) is a matrix obtained by multiplying each entry of row \( r \) of \( A \) by a scalar \( k \). Comparing the cofactor expansion of \( B \) along row \( r \) to that of \( A \), it is easy to see that \( \det B = k \cdot \det A \). We leave the details to the reader.

(c) We first show that if \( C \) is an \( n \times n \) matrix having two identical rows, then \( \det C = 0 \). Suppose that rows \( r \) and \( s \) of \( C \) are equal, and let \( M \) be obtained from \( C \) by interchanging rows \( r \) and \( s \). Then \( \det M = -\det C \) by (a). But since rows \( r \) and \( s \) of \( C \) are equal, we also have \( C = M \). Thus \( \det C = \det M \). Combining the two equations involving \( \det M \), we obtain \( \det C = -\det C \). Therefore \( \det C = 0 \).

Now suppose that \( B \) is obtained from \( A \) by adding \( k \) times row \( s \) of \( A \) to row \( r \), where \( r \neq s \). Let \( C \) be the \( n \times n \) matrix obtained from \( A \) by replacing \( a'_r = (u_1, u_2, \ldots, u_n) \) by \( a'_r = (v_1, v_2, \ldots, v_n) \). Since \( A, B, \) and \( C \) differ only in row \( r \), we have \( A_{ij} = B_{ij} = C_{ij} \) for every \( j \). Using the cofactor expansion of \( B \) along row \( r \), we obtain

\[
\det B = (u_1 + kv_1)(-1)^{r+1} \det B_{1 \cdot} + \cdots + (u_n + kv_n)(-1)^{n+1} \det B_{n \cdot} \\
= (u_1(-1)^{r+1} \det B_{1 \cdot} + \cdots + u_n(-1)^{n+1} \det B_{n \cdot}) \\
\quad + k (v_1(-1)^{r+1} \det B_{1 \cdot} + \cdots + v_n(-1)^{n+1} \det B_{n \cdot}) \\
= [u_1(-1)^{r+1} \det A_{1 \cdot} + \cdots + u_n(-1)^{n+1} \det A_{n \cdot}] \\
\quad + k [v_1(-1)^{r+1} \det C_{1 \cdot} + \cdots + v_n(-1)^{n+1} \det C_{n \cdot}].
\]

In this equation, the first expression in brackets is the cofactor expansion of \( A \) along row \( r \), and the second is the cofactor expansion of \( C \) along row \( r \). Thus we have

\[
\det B = \det A + k \cdot \det C.
\]

However, \( C \) is a matrix with two identical rows (namely, rows \( r \) and \( s \), which are both equal to \( a'_r(\cdot) \)). Since \( \det C = 0 \) by the preceding paragraph, it follows that \( \det B = \det A \).

(d) Let \( E \) be an elementary matrix obtained by interchanging two rows of \( I_n \). Then \( \det EA = -\det A \) by (a). Since \( \det E = -1 \), we have \( \det EA = (\det E)(\det A) \). Similar arguments establish (d) for the other two types of elementary matrices.

Practice Problems

1. Use elementary row operations to evaluate the determinant of

\[
A = \begin{bmatrix}
1 & 3 & -3 \\
-3 & -9 & 2 \\
-4 & 4 & -6
\end{bmatrix}
\]

2. For what value of \( c \) is

\[
B = \begin{bmatrix}
1 & -1 & 2 \\
-1 & 0 & c \\
2 & 1 & 4
\end{bmatrix}
\]

not invertible?

Exercises

1. Determine if the following statements are true or false.
   (a) The determinant of a square matrix equals the product of its diagonal entries.
   (b) Performing a row addition operation on a square matrix does not change its determinant.
   (c) Performing a scaling operation on a square matrix does not change its determinant.
   (d) Performing an interchange operation on a square matrix changes its determinant by a factor of \(-1\).
   (e) For any \( n \times n \) matrices \( A \) and \( B \), \( \det (A + B) = \det A + \det B \).
(f) For any $n \times n$ matrices $A$ and $B$, $\det AB = (\det A)(\det B)$.

(g) If $A$ is any invertible matrix, then $\det A = 0$.

(h) For any square matrix $A$, $\det A^T = -\det A$.

(i) The determinant of any square matrix can be evaluated using a cofactor expansion along any column.

(j) The determinant of any square matrix equals the product of the diagonal entries of its reduced row echelon form.

In Exercises 2–4, evaluate the determinant of the given matrix using a cofactor expansion along the indicated column.

2. \[
\begin{bmatrix}
1 & -2 & 2 \\
2 & -1 & 3 \\
0 & 1 & -1 \\
\end{bmatrix}
\]
first column

3. \[
\begin{bmatrix}
2 & -1 & 3 \\
1 & 4 & -2 \\
-1 & 0 & 1 \\
\end{bmatrix}
\]
second column

4. \[
\begin{bmatrix}
-1 & 2 & -1 \\
5 & -9 & 2 \\
3 & -1 & 2 \\
\end{bmatrix}
\]
third column

In Exercises 5–18, evaluate the determinant of the given matrix using elementary row operations or cofactor expansion.

5. \[
\begin{bmatrix}
0 & 0 & 5 \\
5 & 3 & 7 \\
4 & -1 & -2 \\
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
-6 & 0 & 0 \\
7 & -9 & 4 \\
8 & -2 & 1 \\
\end{bmatrix}
\]

7. \[
\begin{bmatrix}
1 & -2 & 2 \\
0 & 5 & -1 \\
2 & -4 & 1 \\
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
-2 & 1 & -2 \\
4 & -2 & -1 \\
0 & 3 & 6 \\
\end{bmatrix}
\]

9. \[
\begin{bmatrix}
3 & -2 & 1 \\
0 & 0 & 5 \\
-9 & 4 & 2 \\
\end{bmatrix}
\]

10. \[
\begin{bmatrix}
-2 & 6 & 1 \\
0 & 0 & 3 \\
4 & -1 & 2 \\
\end{bmatrix}
\]

11. \[
\begin{bmatrix}
1 & 4 & 2 \\
2 & -1 & 3 \\
-1 & 3 & 1 \\
\end{bmatrix}
\]

12. \[
\begin{bmatrix}
-1 & 2 & 1 \\
5 & -9 & -2 \\
3 & 1 & 2 \\
\end{bmatrix}
\]

13. \[
\begin{bmatrix}
1 & 2 & 1 \\
1 & 1 & 2 \\
3 & 4 & 8 \\
\end{bmatrix}
\]

14. \[
\begin{bmatrix}
3 & 4 & 2 \\
2 & -1 & 3 \\
-1 & 3 & 1 \\
\end{bmatrix}
\]

15. \[
\begin{bmatrix}
1 & -1 & 2 & 1 \\
2 & -1 & -1 & 4 \\
-4 & 5 & -10 & -6 \\
3 & -2 & 10 & -1 \\
\end{bmatrix}
\]

16. \[
\begin{bmatrix}
2 & 1 & 5 & 2 \\
2 & 1 & 8 & 1 \\
2 & -1 & 5 & 3 \\
4 & -2 & 10 & 3 \\
\end{bmatrix}
\]

17. \[
\begin{bmatrix}
0 & 4 & -1 & 1 \\
-3 & 1 & 1 & 2 \\
1 & 0 & -2 & 3 \\
2 & 3 & 0 & 1 \\
\end{bmatrix}
\]

18. \[
\begin{bmatrix}
1 & -1 & 2 & -1 \\
2 & -2 & -3 & 8 \\
-3 & 4 & 1 & -1 \\
-2 & 6 & -4 & 18 \\
\end{bmatrix}
\]

For each of the matrices in Exercises 19–28, determine the value(s) of $c$ for which the given matrix is not invertible.

19. \[
\begin{bmatrix}
4 & c \\
3 & -6 \\
\end{bmatrix}
\]

20. \[
\begin{bmatrix}
3 & 9 \\
5 & c \\
\end{bmatrix}
\]

21. \[
\begin{bmatrix}
c & 6 \\
2 & c + 4 \\
\end{bmatrix}
\]

22. \[
\begin{bmatrix}
c & c - 1 \\
-8 & c - 6 \\
\end{bmatrix}
\]

23. \[
\begin{bmatrix}
0 & -1 \\
3 & 4 \\
\end{bmatrix}
\]

24. \[
\begin{bmatrix}
1 & 2 & -6 \\
2 & 4 & c \\
-3 & -5 & 7 \\
\end{bmatrix}
\]

25. \[
\begin{bmatrix}
1 & -1 & 2 \\
-2 & 2 & 4 \\
1 & -1 & 0 \\
\end{bmatrix}
\]

26. \[
\begin{bmatrix}
1 & 2 & c \\
-2 & -2 & 4 \\
1 & 6 & -12 \\
\end{bmatrix}
\]

27. \[
\begin{bmatrix}
2 & c \\
3 & c \\
0 & c - 15 \\
\end{bmatrix}
\]

28. \[
\begin{bmatrix}
-1 & 1 & 1 \\
3 & -2 & -c \\
0 & c & -10 \\
\end{bmatrix}
\]

In Exercises 29–36, solve the given system using Cramer's rule.

29. \[
\begin{align*}
x_1 + 2x_2 &= 6 \\
x_2 &= 6 \\
3x_1 + 4x_2 &= 7 \\
3x_1 + 4x_2 &= 6 \\
7x_1 + 12x_2 &= 5 \\
6x_1 + 5x_2 &= 9 \\
-2x_1 + x_2 + 3x_3 &= -5 \\
x_2 + x_3 &= 4 \\
-x_1 + x_2 + x_3 &= -3 \\
x_2 + 2x_3 &= -1 \\
x_1 - x_2 + 3x_3 &= 4 \\
-2x_1 - x_2 + x_3 &= -2 \\
3x_1 + x_2 - x_3 &= 1 \\
-x_1 + 2x_2 + x_3 &= -1 \\
\end{align*}
\]

37. Give an example to show that $\det kA \neq k \cdot \det A$ for some matrix $A$ and scalar $k$.

38. Evaluate $\det kA$ if $A$ is an $n \times n$ matrix and $k$ is a scalar. Justify your answer.

39. Prove that if $A$ is an invertible matrix, then

$$\det A^{-1} = \frac{1}{\det A}.$$
3.2. Properties of Determinants

52. Let $A$ be an $n \times n$ matrix and $c_{jk}$ denote the $(k, j)$-cofactor of $A$.

(a) Prove that if $B$ is the matrix obtained from $A$ by replacing column $k$ by $e_j$, then $det B = c_{kj}$.

(b) Show that for each $j$ we have

$$\begin{bmatrix}
    c_{1j} \\
    c_{2j} \\
    \vdots \\
    c_{nj}
\end{bmatrix} = \det A \cdot e_j.$$

*Hint:* Apply Cramer’s rule to $Ax = e_j$.

(c) Deduce that if $C$ is the $n \times n$ matrix whose $(i, j)$-entry is $c_{ij}$, then $AC = (\det A)I_n$. This matrix $C$ is called the classical adjoint of $A$.

(d) Show that if $\det A \neq 0$, then $A^{-1} = \frac{1}{\det A} C$.

In Exercises 53–55, use a calculator with matrix capabilities or computer software such as MATLAB to solve the problem.

53. (a) Use elementary row operations other than scaling operations to transform

$$A = \begin{bmatrix}
0 & -3.0 & -2 & -5 \\
2.4 & 3.0 & -6 & 9 \\
-4.8 & 6.3 & 4 & -2 \\
9.6 & 1.5 & 5 & 9
\end{bmatrix}$$

into an upper triangular matrix $U$.

(b) Use the boxed result on page 187 to compute $\det A$.

54. (a) Solve $Ax = b$ using Cramer’s rule, where

$$A = \begin{bmatrix}
    0 & 1 & 2 & -1 \\
    1 & 2 & 1 & -2 \\
    2 & -1 & 0 & 3 \\
    3 & 0 & -3 & 1
\end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix}
    24 \\
    -16 \\
    8 \\
    10
\end{bmatrix}.$$

(b) How many determinants of $4 \times 4$ matrices are evaluated in (a)?

55. Compute the classical adjoint (as defined in Exercise 52) of the matrix $A$ in Exercise 54.

50. The following sequence of elementary row operations transforms $A$ into an upper triangular matrix $U$.

$$\begin{bmatrix}
1 & 3 & -3 \\
-3 & -9 & 2 \\
-4 & 4 & -6
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 3 & -3 \\
0 & 0 & -7 \\
0 & 16 & -18
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 3 & -3 \\
0 & 16 & -18 \\
0 & 0 & -7
\end{bmatrix} = U$$

Since one row interchange operation was performed, we have

$$\det A = (-1)^1 \cdot \det U = (-1)(1)(16)(-7) = 112.$$