

The Linear Chromatic Number of Sperner families

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Abstract

Let S be a set with m elements and \mathcal{S} a complete Sperner family on S , i.e. a Sperner family such that every $x \in S$ is contained in some member of \mathcal{S} . The linear chromatic number of \mathcal{S} , defined by Civan, is the smallest integer n with the property that there exists a function $f : S \rightarrow \{1, \dots, n\}$ such that if $f(x) = f(y)$, then every set in \mathcal{S} which contains x also contains y , or every set in \mathcal{S} which contains y also contains x . We give an explicit formula for the number of complete Sperner families of linear chromatic number 2. We also prove tight bounds on the number of elements in a Sperner family of given chromatic number, and prove that complete Sperner families of maximum linear chromatic number far outnumber those of lesser linear chromatic number.

1 Introduction

Let S be a set. A *Sperner family* on S is antichain in the poset 2^S ; that is, a family \mathcal{S} of subsets of S which are pairwise incomparable by inclusion. For $x \in S$, write $\mathcal{S}_x = \{A \in \mathcal{S} : x \in A\}$; \mathcal{S} is called *complete* if $\bigcup_{A \in \mathcal{S}} A = S$, or equivalently $\mathcal{S}_x \neq \emptyset$ for all $x \in S$. Sperner families have long been of interest in extremal set theory,

beginning in the 1930s with the seminal work of Sperner [7] on the maximum size of such a family.

The problem of enumerating the Sperner families on a given set actually dates as far back as Dedekind and is equivalent to that of enumerating the free distributive lattice on n generators. This problem has been studied extensively, and, while an exact formula seems elusive at best, there are asymptotic formulas for this quantity; see [6] for a comprehensive survey of work to date on this problem.

An elementary but useful observation is that there is a bijection between ideals (or filters) in a poset and antichains in the same poset. Any antichain generates an ideal in the natural way; conversely, the set of maximal elements in an ideal forms an antichain. A *simplicial complex* on a set S is an ideal in 2^S which contains all the singleton subsets of S . Applied to the poset 2^S , it is easy to see that the above-mentioned bijection restricts to a bijection between simplicial complexes on S and complete Sperner families on S .

In [2], Civan and Yalcin defined and studied the notion of a linear coloring of a simplicial complex, the motivation coming from the study of multicomplexes. Using the above bijection, Civan [1] defined linear colorings for complete Sperner families: given a set S and integer $k > 0$, a *linear k -coloring* of a complete Sperner family \mathcal{S} is a function $f : S \rightarrow [k] = \{1, \dots, k\}$ such that if $f(x) = f(y)$, then $\mathcal{S}_x \subseteq \mathcal{S}_y$ or $\mathcal{S}_y \subseteq \mathcal{S}_x$. This is equivalent to requiring that for any $i \in [1, k]$, there is a linear ordering (by refinement) of the set of families $\{\mathcal{S}_v : f(v) = i\}$. The sets $f^{-1}(m)$ are called *color classes*. The Sperner family \mathcal{S} is called *linearly k -colorable* if such a k -coloring exists, and the *linear chromatic number* of \mathcal{S} , denoted $\lambda(\mathcal{S})$, is the smallest $k > 0$ for which \mathcal{S} is k -linearly colorable. We say that elements $x, y \in S$ are *separated* by S if there exist $A, B \in \mathcal{S}$ such that $x \in A$, $y \notin A$, $y \in B$, and $x \notin B$; in any linear coloring of \mathcal{S} , elements separated by S must lie in different color classes.

Following [1], we write $ls(m; n)$ for the number of complete Sperner families of linear chromatic number n on a set of size m and $LS(m; n)$ for the set of all such families; we also write $LS(m) = \bigcup_{k=1}^m LS(m; k)$. It is clear that for any $m > 0$, $ls(m; 1) = 1$ and that $ls(m; n) = 0$ when $n > m$. When considering Sperner families on a particular set S , we also use notation such as $LS(m; S)$, $LS(S)$ with the same meaning as above.

The purpose of this article is to address several of the questions raised by Civan in [1]. Section 2 establishes a general framework within which the main results of the paper are proved. In Section 3, we give a formula for $ls(m; 2)$. In Section 4, we give (tight) bounds on the size of a Sperner family of specified linear chromatic number.

Finally, in Section 5 we prove that in general, Sperner families of maximum linear chromatic number far outnumber those of lesser chromatic number.

We now introduce terminology associated with posets which will be used later in the paper; the reader may wish to consult [4] for a more comprehensive treatment. If (P, \leq) is a poset and $x, y \in P$, we say that y covers x if $x \leq y$ and if $x \leq z \leq y$ then $z = x$ or $z = y$. A *subposet* of a poset (P, \leq) is described by a subset $S \subseteq P$ together with the inherited partial order. A k -family in P is a subset $S = \{x_1, \dots, x_n\} \subseteq P$ such that S contains no chain of length $k + 1$; in particular, a 1-family is merely an antichain. P is called *ranked* (or *graded*) if there exists a *ranking function* $r : P \rightarrow \mathbb{Z}$ such that $r(y) = r(x) + 1$ whenever y covers x . Given such a ranking function, we define the i th *level* of P to be the set of elements of P of rank i . If the ranking function has range $[0, m]$ and L_i denotes the i th level, we say that P is *rank-symmetric* if $|L_i| = |L_{m-i}|$ for all i and *rank-unimodal* if there exists $\mu \in [0, m]$ such that $|L_i| \leq |L_j|$ for $0 \leq i \leq j \leq \mu$ and $|L_i| \geq |L_j|$ for $\mu \leq i \leq j \leq m$. Finally, P has the *strong Sperner property* if for all k , the maximum size of a k -family in P is equal to the sum of the sizes of the k largest levels in P ; P has the *Sperner property* if the maximum size of an antichain in P is equal to the size of the largest level. Note also that we may illustrate a poset P by a directed graph with underlying set P and an arc from x to y if and only if y covers x . We use the notation $d_+(x)$ (respectively $d_-(x)$) to denote the out-degree (respectively, in-degree) of x ; these correspond (in the poset) to the number of elements which cover (respectively, are covered by) x .

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2 Linear Chromatic Number and Chain Products

Let S be a set and $\mathcal{S} = \{A_1, \dots, A_k\}$ a Sperner family on S . For each $x \in S$, let $i(x) = \{i : x \in A_i\}$. Then the sets $\mathcal{A}(\mathcal{S}) = \{i(x) : x \in S\}$ induce a subposet of the Boolean lattice $2^{[k]}$. We then have the following alternate characterization of linear chromatic number:

Proposition 2.1. $\lambda = \lambda(\mathcal{S})$ is the smallest ℓ such that $\mathcal{A}(\mathcal{S})$ can be partitioned into ℓ chains.

For any positive integer n , we denote by L_n the (finite) linear order with underlying set $\{1, \dots, n\}$ and the obvious order. Likewise, we denote by \tilde{L}_n the linear order with underlying set $\{0, 1, \dots, n\}$. Now let n_1, \dots, n_r be positive integers. A poset of the

form $C_{n_1, \dots, n_r} = L_{n_1} \times \dots \times L_{n_r}$ is called a *chain product*; we use the notation $\tilde{C}_{n_1, \dots, n_r}$ to mean $\tilde{L}_{n_1} \times \dots \times \tilde{L}_{n_r}$.

Note that $\tilde{C}_{n_1, \dots, n_r}$ has a natural structure as a ranked poset via the function $r(a_1, \dots, a_r) = a_1 + \dots + a_r$. We record the following useful fact.

Proposition 2.2. [4, p. 230] *Chain products are rank-symmetric, rank-unimodal, and have the strong Sperner property.*

Chain products are intimately related to the notion of linear colorings by the following construction. Given a finite set S and a partition $\mathcal{P} = \{S_1, \dots, S_n\}$ of S , a *ranking* of \mathcal{P} is a collection $R = \{r_i\}_{i=1}^n$ of bijective functions $r_i : S_i \rightarrow \{1, \dots, |S_i|\}$, $i = 1, \dots, n$. Now for a (finite) set S , let $M_n(S)$ denote the set of ordered triples $(\mathcal{P}, R, \mathcal{C})$ where $\mathcal{P} = \{S_1, \dots, S_n\}$ is a partition of S , $R = \{r_i : i = 1, \dots, n\}$ is a ranking of \mathcal{P} , and \mathcal{C} is antichain in $\tilde{C}_{|S_1|, \dots, |S_n|}$. Then there is a function

$$\Phi : M_n(S) \rightarrow \cup_{k \leq n} LS(m; k)$$

defined as follows: given a triple $t = (\mathcal{P}, R, \mathcal{C}) \in M_n(S)$, each element $c \in \mathcal{C}$ corresponds to an n -tuple (c_1, \dots, c_n) , where for each i , $0 \leq c_i \leq |S_i|$. Now define $c_S = \bigcup_{i=1}^n \{s_i \in S_i : r_i(s_i) \leq c_i\}$. It is easy to check that $\Phi(t) = \{c_S : c \in \mathcal{C}\}$ is an n -linearly colorable Sperner family in S , where the color classes are S_1, \dots, S_n .

Proposition 2.3. Φ is surjective; moreover, if $\Phi(\mathcal{P}, R, \mathcal{C}) = \mathcal{S}$, then $|\mathcal{C}| = |\mathcal{S}|$.

Proof.

Let \mathcal{S} be an n -linearly colorable Sperner family on S . Then there is a partition of \mathcal{P} of S into color classes S_1, \dots, S_n , and for each $i = 1, \dots, n$ there exists a bijection $r_i : S_i \rightarrow [1, \dots, |S_i|]$ such that $\mathcal{S}_x \subseteq \mathcal{S}_y$ if and only if $r_i(y) \leq r_i(x)$. Now consider the element of $M_n(S)$ described by the partition \mathcal{P} , the ranking $R = \{r_i\}_{i=1}^n$, and the antichain $\mathcal{C} = \{C(A) : A \in \mathcal{S}\}$ in $\tilde{C}_{|S_1|, \dots, |S_n|}$, where $C(A) = (c_1, \dots, c_n)$ is defined by $c_i = \max\{0\} \cup r_i(A \cap S_i)$. Clearly $\Phi(\mathcal{P}, R, \mathcal{C}) = \mathcal{S}$, and the equality $|\mathcal{C}| = |\mathcal{S}|$ is obvious from the construction. \square

Lemma 2.4. *There is a bijection*

$$\alpha : \{\text{antichains in } \tilde{C}_{n_1, n_2}\} \rightarrow \{(X, Y) : X \subseteq \tilde{L}_{n_1}, Y \subseteq \tilde{L}_{n_2}, |X| = |Y|\}$$

Proof.

If $A = \{(x_1, y_1), \dots, (x_m, y_m)\}$ is an antichain in \tilde{C}_{n_1, n_2} , then x_1, \dots, x_m are distinct, as are y_1, \dots, y_m . We associate to A the pair (X, Y) where $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_m\}$. Conversely, given such a pair (X, Y) , write $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_m\}$ where $x_1 < \dots < x_m$ and $y_1 < \dots < y_m$, and associate to it the antichain $\{(x_1, y_m), (x_2, y_{m-1}), \dots, (x_m, y_1)\}$; these operations are clearly inverse to each other. \square

There is a type of duality operation which is often useful for constructing examples and is closely related to the linear chromatic number. It is defined by sending a family $\mathcal{F} = \{A_i\}_{i \in I}$ of subsets of a set S to the collection $\mathcal{F}^\vee = \{B_x : x \in S\}$ of subsets of I , where $B_x = \{i \in I : x \in A_i\}$. The next result follows immediately from the definitions:

Proposition 2.5.

- For all \mathcal{F} , $(\mathcal{F}^\vee)^\vee = \mathcal{F}$
- Let $\mathcal{S} = \{A_i\}_{i \in I}$ be a Sperner family on a set S of cardinality m . Then \mathcal{S}^\vee is a Sperner family on I if and only if $\lambda(\mathcal{S}) = m$.

Lemma 2.6. Let S be a set of cardinality m and \mathcal{S} a Sperner family on S . Then $\lambda(\mathcal{S}) = m$ if and only if \mathcal{S} separates every pair of (distinct) vertices $a, b \in S$.

Proof.

If a and b are vertices of S and \mathcal{S} does not separate a from b , then without loss of generality $\mathcal{S}_a \subseteq \mathcal{S}_b$. By assigning one color to a and b and distinct colors to the other vertices, we see that $\lambda(\mathcal{S}) \leq m - 1$. Conversely, if $\lambda(\mathcal{S}) \leq m - 1$, then there exist vertices $a, b \in S$ such that $\mathcal{S}_a \subseteq \mathcal{S}_b$, and so \mathcal{S} does not separate a from b . \square

Let S be a set and \mathcal{S} a Sperner family on S . We define the set of *base points* of \mathcal{S} to be the subset of elements of S contained in every set in the family \mathcal{S} ; that is, $B(\mathcal{S}) = \{x \in S : \mathcal{S}_x = \mathcal{S}\}$. If $B(\mathcal{S}) = \emptyset$, \mathcal{S} is called *primitive*.

Primitivity and completeness are dual concepts, as demonstrated by the following:

Proposition 2.7. Let S be a set and \mathcal{S} a Sperner family on S . Then $\mathcal{S}^C = \{S - A : A \in \mathcal{S}\}$ is also a Sperner family on S . Moreover, \mathcal{S} is complete if and only if \mathcal{S}^C is primitive.

Proof.

Since no two elements of \mathcal{S} are comparable by inclusion, the same is true of \mathcal{S}^C ; hence the latter is a Sperner family on S . Now \mathcal{S} is complete if and only if $\bigcup_{A \in \mathcal{S}} A = S$; this

is equivalent (by complementation) to $\bigcap_{B \in \mathcal{S}^c} B = S - (\bigcup_{B \in \mathcal{S}^c} (S - B)) = S - (\bigcup_{A \in \mathcal{S}} A) = \emptyset$. \square

There is another approach to the study of Sperner families which ties together several of the concepts introduced above and involves the diameter of a directed graph obtained (from the Sperner family) by orienting the edges of a complete bipartite graph. Such objects are of independent interest in the literature; see for example [8].

Let S be a set with m elements and \mathcal{F} a family of subsets of S . We define the directed graph $D = D(S, \mathcal{F})$ (which is an orientation of $K_{|S|, |\mathcal{F}|}$) as follows. Identifying the partite sets with S and \mathcal{F} respectively, there is an arc from $x \in S$ to $A \in \mathcal{F}$ if and only if $x \in A$; otherwise, there is an arc from A to x . Then we have the following:

- Proposition 2.8.** *1. \mathcal{F} is a Sperner family \Leftrightarrow every pair of vertices $A, B \in \mathcal{F}$ is contained in a 4-cycle of D .*
- 2. \mathcal{F} is complete $\Leftrightarrow S \subseteq V(D)$ contains no sink.*
- 3. \mathcal{F} is primitive $\Leftrightarrow S \subseteq V(D)$ contains no source.*
- 4. If \mathcal{F} is a complete linear Sperner family, then $\lambda(\mathcal{F}) = m \Leftrightarrow$ there is a directed path between any pair of (distinct) vertices $x, y \in S \subseteq V(D) \Leftrightarrow D$ has diameter 3.*

Proof.

For the first statement, note that the condition that the sets A and B in \mathcal{F} be pairwise incomparable is equivalent to the existence of elements such that $x \in A \setminus B, y \in B \setminus A$, or equivalently the existence of a cycle in D of the form $x \rightarrow A \rightarrow y \rightarrow B \rightarrow x$. The second and third statements follow directly from the definitions.

For the fourth statement, note first that for any $x \in S$ and $y \in \mathcal{F}$, exactly one of $x \rightarrow y, y \rightarrow x$ is *not* an arc in D , so D must have diameter at least 3. Now Lemma 2.6 implies that $\lambda(\mathcal{F}) = m$ is equivalent to the condition that \mathcal{F} separates every pair of distinct vertices $x, y \in S$. In terms of the directed graph D , this is precisely the existence of sets A and B in \mathcal{F} such that $x \in A \setminus B$ and $y \in B \setminus A$, or equivalently that there is a path in D of length two between any such x and y . Now suppose \mathcal{F} separates every pair of distinct vertices in S . To show that D has diameter 3, it remains to verify that there is a directed path of length at most 3 between any $x \in S$ and $A \in \mathcal{S}$, and also a path from A to x . If $x \in A$, $x \rightarrow A$ is an arc, and we may construct a path from A back to x by selecting $y \notin A$ and choosing a set $B \in \mathcal{F}$ which contains y but not x . If $x \notin A$, $A \rightarrow x$ is an arc, and we may construct a path from x to A by

selecting $y \notin A$ and a set $B \in \mathcal{F}$ which contains x but not y . Finally, suppose \mathcal{F} does not separate vertices. Then there exist $x, y \in S$ such that $\mathcal{F}_x \subseteq \mathcal{F}_y$. This implies that there is no path of length 2 from x to y in D ; since D is an orientation of a bipartite graph, this forces the diameter of D to be at least 4. \square

3 On $ls(m; 2)$

In this section we give a general formula for $ls(m; 2)$. We denote by $ls_p(m; n)$ the number of complete primitive Sperner families of linear chromatic number n on a set of cardinality m .

Lemma 3.1. *Let S be a set and $T \subseteq S$ a subset. Then there is a bijective correspondence:*

$$h : \{\mathcal{S} \in LS(S) : B(\mathcal{S}) = T\} \rightarrow \{\mathcal{T} \in LS(S - T) : \mathcal{T} \text{ is primitive}\}$$

Moreover, h preserves linear chromatic number; that is, if $B(\mathcal{S}) = T$ and $\mathcal{S} \in LS(S; k)$, then $h(\mathcal{S}) \in LS(S - T; k)$.

Proof.

Given a Sperner family \mathcal{S} on S with $B(\mathcal{S}) = T$, it is easy to check that $h(\mathcal{S}) = \{A - T : A \in \mathcal{S}\}$ is a complete primitive Sperner family on $S - T$ and that an inverse function is given by sending a complete primitive Sperner family \mathcal{T} on T to $j(\mathcal{T}) = \{A \cup T : A \in \mathcal{T}\}$, which is a complete Sperner family on S whose set of base points is exactly T . Now suppose $f : S \rightarrow \{1, \dots, k\}$ is a k -linear coloring of \mathcal{S} ; that is, for each i , $\{\mathcal{S}_x : f(x) = i\}$ is linearly ordered by refinement. Then $\{[h(\mathcal{S})]_x : x \in S - T, f(x) = i\}$ is a set of families of subsets of $S - T$, also linearly ordered by refinement. This implies that $f|_{S-T} : S - T \rightarrow \{1, \dots, k\}$ is a k -linear coloring of $h(\mathcal{S})$, and so $\lambda(h(\mathcal{S})) \leq \lambda(\mathcal{S})$. Conversely, if $g : S - T \rightarrow \{1, \dots, k\}$ is a k -linear coloring of a complete primitive Sperner family \mathcal{T} on $S - T$, then by extending g in any way to a function $f : S \rightarrow \{1, \dots, k\}$, we obtain a k -linear coloring of $j(\mathcal{T})$; this is so because $[j(\mathcal{T})]_x = j(\mathcal{T})$ for any $x \in T$. Hence, $\lambda(j(\mathcal{T})) \leq \lambda(\mathcal{T})$. Since j and h are inverse to each other, the claim is proven. \square

The next result permits us to reduce the calculation of $ls(m; n)$ to that of $ls_p(m; n)$. It follows immediately from Lemma 3.1 and the fact that $ls_p(k; n) = 0$ when $k < n$.

Corollary 3.2. $ls(m; n) = \sum_{k=n}^m \binom{m}{k} ls_p(k; n)$.

We now turn to the correspondence Φ of Section 2 and explain how it may be used to compute $ls(m; 2)$. To facilitate the discussion, we introduce several notions. For any nonempty subset $S \subseteq [k]$, write $S = \{a_1, \dots, a_l\}$ where $a_1 < \dots < a_l$, and define the *elasticity* of S by $e(S) = a_1! \prod_{i=1}^{l-1} (a_{i+1} - a_i)!$.

Now suppose S is a set of cardinality m and \mathcal{S} is a complete Sperner family on S such that $\lambda(\mathcal{S}) = 2$. This means that there exists a partition of S into nonempty subsets S_1 and S_2 , together with an ordering \preceq_i on S_i $i = 1, 2$, such that $x \preceq_i y$ if and only if $\mathcal{S}_y \subseteq \mathcal{S}_x$. We study the fiber $\Phi^{-1}(\mathcal{S})$.

Lemma 3.3. *Let S be a set and \mathcal{S} a Sperner family on S of linear chromatic number 2, and suppose $\Phi(\mathcal{P}, R, \mathcal{C}) = \mathcal{S}$.*

- *If \mathcal{S} is complete, then $(|S_1|, b) \in \mathcal{C}$ for some b , $0 \leq b \leq |S_2|$ and $(a, |S_2|) \in \mathcal{C}$ for some a , $0 \leq a \leq |S_1|$.*
- *If \mathcal{S} is primitive, then $(0, b) \in \mathcal{C}$ for some b , $0 \leq b \leq |S_2|$ and $(a, 0) \in \mathcal{C}$ for some a , $0 \leq a \leq |S_1|$.*
- *If \mathcal{S} is both complete and primitive, then $(0, |S_2|), (|S_1|, 0) \in \mathcal{C}$ and $S_1, S_2 \in \mathcal{S}$.*

Proof.

If $(|S_1|, b) \notin \mathcal{C}$ for all b , $0 \leq b \leq |S_2|$, then (by construction of Φ), no subset in \mathcal{S} contains $r_1^{-1}(|S_1|)$ and \mathcal{S} is not complete; a similar argument establishes that $(a, |S_2|) \in \mathcal{C}$ for some a if \mathcal{S} is complete. If $(0, b) \notin \mathcal{C}$ for all b , $0 \leq b \leq |S_2|$, then $r_1^{-1}(1)$ is a base point of \mathcal{S} and so \mathcal{S} is not primitive; a similar argument shows that $(a, 0) \in \mathcal{C}$ for some a if \mathcal{S} is primitive. If \mathcal{S} is both complete and primitive, then the first two statements together with the fact that \mathcal{C} is an antichain force $(0, |S_2|), (|S_1|, 0) \in \mathcal{C}$. By construction of Φ , $S_1, S_2 \in \mathcal{S}$.

Lemma 3.4. *Let S be a set and \mathcal{S} a complete primitive Sperner family on S of linear chromatic number 2. Then the two color classes are uniquely determined and are members of \mathcal{S} .*

Proof.

Fix a linear coloring $f : S \rightarrow \{1, 2\}$; let $S_1 = f^{-1}(1) = \{x_1, \dots, x_p\}$, $S_2 = f^{-1}(2) = \{y_1, \dots, y_q\}$, where $\mathcal{S}_{x_i} \subseteq \mathcal{S}_{x_j}$ and $\mathcal{S}_{y_i} \subseteq \mathcal{S}_{y_j}$ for all $i < j$. We claim that the sets S_1 and S_2 are members of \mathcal{S} . Since \mathcal{S} is complete, there exist sets $A, B \in \mathcal{S}$ such that $x_p \in A$ and $y_q \in B$. Suppose towards a contradiction that every such A contains y_1 . If some $C \in \mathcal{S}$ contains none of y_1, \dots, y_q , then C is contained in any such subset A ,

violating the assumption that \mathcal{S} is a Sperner family. Otherwise, all $C \in \mathcal{S}$ contain y_1 , and so y_1 is a base point for \mathcal{S} , contradicting primitivity of \mathcal{S} . Hence $S_1 \in \mathcal{S}$, and a similar argument establishes that $S_2 \in \mathcal{S}$. It is now clear that \mathcal{S} separates every element of S_1 from S_2 , and hence that any 2-linear coloring of \mathcal{S} must assign all elements of S_1 to one color class and all elements of S_2 to the other. \square

Corollary 3.5. *Suppose \mathcal{S} is a complete primitive Sperner family of linear chromatic 2 on a set S and that $\Phi(\mathcal{P}, R, \mathcal{C}) = \mathcal{S}$. Then \mathcal{P} and \mathcal{C} are uniquely determined.*

Proof.

Uniqueness of \mathcal{P} follows from Lemma 3.4. To see that \mathcal{C} is uniquely determined, first note that $|\mathcal{C}| = |\mathcal{S}|$ is uniquely determined. We then prove by induction on $r = |\mathcal{C}| = |\mathcal{C}'|$ that if $\Phi(\mathcal{P}, R, \mathcal{C}) = \Phi(\mathcal{P}, R', \mathcal{C}') = \mathcal{S}$, then $\mathcal{C} = \mathcal{C}'$. If $r = 1$, then $\mathcal{C} = \{(x, y)\}$ and $\mathcal{C}' = \{(x', y')\}$ for some x, x', y, y' , $0 \leq x, x' \leq |S_1|$, $0 \leq y, y' \leq |S_2|$. Let $\mathcal{P} = \{S_1, S_2\}$. Then \mathcal{S} is a one-element Sperner family consisting of a single set A . On one hand, since $\Phi(\mathcal{P}, R, \mathcal{C}) = \{A\}$, $A = A_1 \cup A_2$, where $A_1 = \{r_1^{-1}(0), \dots, r_1^{-1}(x)\} \subseteq S_1$ and $A_2 = \{r_2^{-1}(0), \dots, r_2^{-1}(y)\}$; on the other hand, since $\Phi(\mathcal{P}, R', \mathcal{C}') = \{A\}$, we likewise have $A = A'_1 \cup A'_2$, where $A'_1 = \{r_1'^{-1}(0), \dots, r_1'^{-1}(x')\} \subseteq S_1$ and $A_2 = \{r_2'^{-1}(0), \dots, r_2'^{-1}(y')\}$. Since $S_1 \cap S_2 = \emptyset$, cardinality considerations force $x = x'$ and $y = y'$, and so $\mathcal{C} = \mathcal{C}'$. Now suppose $r > 1$, and write $\mathcal{C} = \{(x_1, y_1), \dots, (x_r, y_r)\}$, $\mathcal{C}' = \{(x'_1, y'_1), \dots, (x'_r, y'_r)\}$, where $x_1 < \dots < x_r$ and $x'_1 < \dots < x'_r$. Notice that since \mathcal{C} and \mathcal{C}' are antichains, this forces $y_1 > \dots > y_r$ and $y'_1 > \dots > y'_r$. We claim that $x_1 = x'_1$ and $y_1 = y'_1$. By construction of Φ , it follows that $|B(\mathcal{S}) \cap S_1| = x_1 = x'_1$. Moreover, the set $A \in \mathcal{S}$ such that $|A \cap S_1|$ is minimized satisfies $|A \cap S_1| = x_1 = x'_1$; so in particular, $|A \cap S_2| = y_1 = y'_1$, as desired. Then $\mathcal{D} = \mathcal{C} - \{(x_1, y_1)\}$ and $\mathcal{D}' = \mathcal{C}' - \{(x'_1, y'_1)\}$ are antichains of size $r - 1$ with $\Phi(\mathcal{P}, R, \mathcal{D}) = \Phi(\mathcal{P}, R', \mathcal{D}') = \mathcal{S} - \{A\}$, so by induction $\mathcal{D} = \mathcal{D}'$. Thus, $\mathcal{C} = \mathcal{C}'$, as desired. \square

Lemma 3.6. *Suppose $t = (\mathcal{P}, R, \mathcal{C}) \in M_2(S)$ and for some $v, w \in S_i$, $1 \leq i \leq 2$, $r_i(v) \leq r_i(w)$. Let $\alpha(\mathcal{C}) = (X_1, X_2)$. Then $[\Phi(t)]_w \subseteq [\Phi(t)]_v$, with equality if and only if $[v, w] \cap X_i = \emptyset$.*

Proof.

For simplicity, we assume $i = 1$; the argument for the other case is identical. For the first statement, $[\Phi(t)]_w = \{S \in \mathcal{S} : w \in S\} = \{c_S : c \in \mathcal{C}, w \in c_S\} = \{c_S : c = (c_1, c_2) \in \mathcal{C} : r_1(w) \leq r_1(c)\} \subseteq \{c_S : c = (c_1, c_2) \in \mathcal{C} : r_1(v) \leq r_1(c)\} = [\Phi(t)]_v$. Equality occurs if and only if there is no $c \in \mathcal{C}$ such that $r_1(v) \leq r_1(c) < r_1(w)$, which is equivalent to $[v, w] \cap X_1 = \emptyset$. \square

Lemma 3.7. *Suppose $\mathcal{P} = \{S_1, S_2\}$ is a partition of S , R is a ranking of \mathcal{P} , and \mathcal{C} is an antichain in $\tilde{\mathcal{C}}_{|S_1|, |S_2|}$. Let $\alpha(\mathcal{C}) = (X, Y)$. For $i \in \{1, 2\}$ and j , $0 \leq j < |S_i|$, define $R' = \{r'_1, r'_2\}$ to be the ranking of \mathcal{P} such that $r'_k = r_k$ for $k \neq i$, $r'_i(x) = r_i(x)$ for $x \notin \{r_i^{-1}(j), r_i^{-1}(j+1)\}$, and $r'_i(r_i^{-1}(j)) = r_i(j+1)$, $r'_i(r_i^{-1}(j+1)) = r_i(j)$. Then $\Phi(\mathcal{P}, R, \mathcal{C}) = \Phi(\mathcal{P}, R', \mathcal{C})$ if and only if $j \notin X \cup Y$.*

Proof.

We give the argument when $i = 1$; the other case follows similarly. For simplicity of notation, we set $s = r_1^{-1}(j)$, $s' = r_1^{-1}(j+1)$. Now suppose $\Phi(\mathcal{P}, R, \mathcal{C}) = \Phi(\mathcal{P}, R', \mathcal{C}) = \mathcal{S}$. By Lemma 3.6, we must have $\mathcal{S}_s \subseteq \mathcal{S}_{s'}$, and so $[j, j+1] \not\subseteq X$; that is, $j \notin X$. Conversely, suppose $j \notin X$ and let $\mathcal{T} = \Phi(\mathcal{P}, R, \mathcal{C})$ and $\mathcal{T}' = \Phi(\mathcal{P}, R', \mathcal{C})$. Then $j \notin X$ implies (by Lemma 3.6) $\mathcal{T}_{r_1^{-1}(j)} = \mathcal{T}_{r_1^{-1}(j+1)}$ and $\mathcal{T}'_{r'_1^{-1}(j)} = \mathcal{T}'_{r'_1^{-1}(j+1)}$. Since the rankings R and R' differ only at the values corresponding to $r_1^{-1}(j)$ and $r_1^{-1}(j+1)$, it follows that $\mathcal{T} = \mathcal{T}'$. \square

Lemma 3.8. *Let S be a set of cardinality m ; let $\mathcal{P} = \{S_1, S_2\}$ be a partition of S and \mathcal{C} an antichain in $\tilde{\mathcal{C}}_{|S_1|, |S_2|}$. Let $\alpha(\mathcal{C}) = (X, Y)$ and $\mathcal{R} = \{R : R \text{ is a ranking of } \mathcal{P}\}$. Let $\mathcal{R}(\mathcal{P}, \mathcal{C}) = \{\Phi(\mathcal{P}, R, \mathcal{C}) : R \in \mathcal{R}, \Phi(\mathcal{P}, R, \mathcal{C}) \text{ is primitive and complete}\}$. Then $|\mathcal{R}(\mathcal{P}, \mathcal{C})| = \frac{|S_1|!|S_2|!}{e(X)e(Y)}$.*

Proof.

Let G be the subgroup of the symmetric group of S which preserves (setwise) the subset S_1 (and hence also S_2). Clearly $|G| = |S_1|!|S_2|!$, and G acts on $\mathcal{R}(\mathcal{P}, \mathcal{C})$ transitively in a natural way. For any particular element $\mathcal{S} \in \mathcal{R}(\mathcal{P}, \mathcal{C})$, Lemma 3.7 implies that the stabilizer $G_{\mathcal{S}}$ is generated by transpositions which exchange elements c and d , where (without loss of generality) either $c, d \in S_1$ and $[c, d] \cap X = \emptyset$ or $c, d \in S_2$ and $[c, d] \cap Y = \emptyset$. Thus $|G_{\mathcal{S}}| = e(X)e(Y)$ and the result follows. \square

Theorem 3.9. *Let S be a set of cardinality $m \geq 2$. Then*

$$\begin{aligned} ls_p(m; 2) &= \frac{m!}{2} \sum_{k=1}^{m-1} \sum_{\substack{A \subseteq [k] \\ k \in A}} \sum_{\substack{B \subseteq [m-k] \\ m-k \in B, |B|=|A|}} \frac{1}{e(A)e(B)} \\ &= m! \left(\sum_{1 \leq k < m/2} \sum_{\substack{A \subseteq [k] \\ k \in A}} \sum_{\substack{B \subseteq [m-k] \\ m-k \in B, |B|=|A|, B \neq A}} \frac{1}{e(A)e(B)} + \frac{1}{2} \sum_{A \subseteq [m/2]: m/2 \in A} \frac{1}{e(A)^2} \right) \end{aligned}$$

Proof.

By Proposition 2.3 and Lemma 3.5, each element of $LS(m; 2)$ is the image under Φ of some triple $(\mathcal{P}, R, \mathcal{C})$, where \mathcal{P} and \mathcal{C} are uniquely determined. Since the Sperner families under consideration are complete and primitive, Lemma 3.3 allows us to conclude that \mathcal{C} contains the elements $(0, |S_2|)$ and $(|S_1|, 0)$. Note that a partition \mathcal{P} consists of an *unordered* pair of (complementary) nonempty subsets of S . Therefore, using Lemma 2.4 to identify antichains with pairs of subsets of equal size, we have:

$$\begin{aligned} l_{s_p}(m; 2) &= \sum_{\mathcal{P}} \sum_{\mathcal{C}} |\mathcal{R}(\mathcal{P}, \mathcal{C})| = \frac{1}{2} \sum_{k=1}^{m-1} \sum_{\substack{A \subseteq [k], k \in A}} \binom{m}{k} \sum_{\substack{B \subseteq [m-k], m-k \in B \\ |B|=|A|}} \frac{k!(m-k)!}{e(A)e(B)} \\ &= \frac{m!}{2} \sum_{k=1}^{m-1} \sum_{\substack{A \subseteq [k] \\ k \in A}} \sum_{\substack{B \subseteq [m-k] \\ m-k \in B, |B|=|A|}} \frac{1}{e(A)e(B)} \end{aligned}$$

□

A quizzical but interesting corollary is the following:

Corollary 3.10. *Let $p \geq 3$ be prime. Then p divides both $l_{s_p}(p; 2)$ and $l_s(p; 2)$.*

Proof.

For $l_{s_p}(p; 2)$, this follows from either formula in Theorem 3.9 and the fact that none of the denominators are divisible by p . For $l_s(p; 2)$ it follows from the result for $l_{s_p}(p; 2)$ and the fact that p divides $\binom{p}{k}$ for $1 \leq k \leq p-1$. □

To illustrate, we compute $l_{s_p}(m; 2)$ and $l_s(m; 2)$ for $m \leq 5$. Obviously, $l_{s_p}(2; 2) = l_s(2; 2) = 1$ and $l_{s_p}(3; 2) = 3$, so by Corollary 3.2, $l_s(3; 2) = 3 + 3 = 6$. Next we calculate $l_{s_p}(4; 2) = 24(1/6 + 1/2 + 1/8) = 19$, and hence $l_s(4; 2) = 19 + 4 \cdot 3 + 6 \cdot 1 = 37$, confirming the calculation in [1], Example 2.8. Finally, $l_{s_p}(5; 2) = 120(1/24 + 1/12 + 1/2 + 1/2) = 135$, so $l_s(5; 2) = 135 + 5 \cdot 19 + 10 \cdot 3 + 10 \cdot 1 = 135 + 95 + 30 + 10 = 270$.

4 On the size of Sperner families

A widely known result of Sperner asserts that the size of a maximum antichain in the Boolean lattice B_m (that is, the maximum size of Sperner family on S), is $\binom{m}{\lfloor m/2 \rfloor}$; this is met by considering the family of subsets of S of size $\lfloor m/2 \rfloor$. A trivial lower bound of 1 is met by the Sperner family $\{S\}$. In this section we address the analogous question, raised in [1], for Sperner families of specified chromatic number. For

convenience, we introduce the following notation. For $m \geq n$, write $m = qn + r$ for some integers q, r , $0 \leq r < n$, and define:

$$s(m, n) = |\{(x_1, \dots, x_n) : \sum_{i=1}^n x_i = \lfloor m/2 \rfloor, 0 \leq x_i \leq \lceil m/n \rceil \text{ for } 1 \leq i \leq r \\ \text{and } 0 \leq x_i \leq \lfloor m/n \rfloor \text{ for } r+1 \leq i \leq n\}|.$$

and

$$t(n) = \min\{t : \binom{t}{\lfloor t/2 \rfloor} \geq n\}.$$

Lemma 4.1. *Suppose $n \geq 2$ and m_1, \dots, m_n are integers such that $\sum_{i=1}^n m_i = m$. Let $s(m_1, \dots, m_n) = |\{(x_1, \dots, x_n) : \sum_{i=1}^n x_i = \lfloor m/2 \rfloor, 0 \leq x_i \leq m_i \text{ for all } i\}|$. If $m_1 \leq \lfloor m/n \rfloor - 1$ and $m_2 \geq \lceil m/n \rceil + 1$, then $s(m_1, \dots, m_n) < s(m_1 + 1, m_2 - 1, m_3, \dots, m_n)$.*

Proof.

If $n = 2$, $s(m_1, m_2) = m_1$ and $s(m_1 + 1, m_2 - 1) = m_1 + 1$, so the claim is true. Now suppose $n \geq 3$. Then

$$\begin{aligned} & s(m_1 + 1, m_2 - 1, m_3, \dots, m_n) - s(m_1, m_2, m_3, \dots, m_n) \\ &= \sum_{j=0}^{m_2-1} |\{(x_3, \dots, x_n) : \sum_{i=3}^n x_i = \lfloor m/2 \rfloor - m_1 - j\}| \\ & \quad - \sum_{j=0}^{m_1} |\{(x_3, \dots, x_n) : \sum_{i=3}^n x_i = \lfloor m/2 \rfloor - m_2 - j\}| \\ &= \sum_{j=m_1}^{m_2-1} |\{(x_3, \dots, x_n) : \sum_{i=3}^n x_i = \lfloor m/2 \rfloor - j\}| - |\{(x_3, \dots, x_n) : \sum_{i=3}^n x_i = \lfloor m/2 \rfloor - m_1 - m_2\}| \end{aligned}$$

Since $m_1 \leq m_2 - 2$, the sum on the far left has at least two terms in it. Thus

$$\begin{aligned} & s(m_1 + 1, m_2 - 1, m_3, \dots, m_n) - s(m_1, m_2, m_3, \dots, m_n) \\ & \geq |\{(x_3, \dots, x_n) : \sum_{i=3}^n x_i = \lfloor m/2 \rfloor - m_1\}| - |\{(x_3, \dots, x_n) : \sum_{i=3}^n x_i = \lfloor m/2 \rfloor - m_1 - m_2\}| \end{aligned}$$

Since $\tilde{C}_{m_3, \dots, m_n}$ is rank-symmetric and rank-unimodal by Proposition 2.2, this difference is positive. \square

Theorem 4.2. *Let S be a set of cardinality m and \mathcal{S} a complete Sperner family on S of linear chromatic number n . Then $t(n) \leq |\mathcal{S}| \leq s(m, n)$ and both bounds are tight.*

Proof.

For the lower bound, suppose that there exists a Sperner family of linear chromatic number n and cardinality t on a set of cardinality m , where $\ell = \binom{t}{\lfloor t/2 \rfloor} < n$. After fixing a bijection between \mathcal{S} and $[t]$, we may identify, for each $x \in \mathcal{S}$, \mathcal{S}_x with a subset $T_x \subseteq [t]$. By a now well-known result [3], there exists a partition of the poset $2^{[t]}$ into disjoint chains C_1, \dots, C_ℓ ; thus, we may construct an ℓ -linear coloring $\lambda : S \rightarrow [\ell]$ of \mathcal{S} by declaring $\lambda(x)$ to be the unique index i such that T_x is a member of C_i . This shows that $\lambda(\mathcal{S}) \leq \ell$. Now consider the family $\mathcal{A} = \{A \subseteq [t(n)] : |A| = \lfloor t(n)/2 \rfloor\} = \{A_1, \dots, A_\ell\}$; this is a complete Sperner family of linear chromatic number $t(n)$, so by Proposition 2.5, \mathcal{A}^\vee is a complete Sperner family of subsets of $[t]$ of linear chromatic number ℓ . Finally, fix a bijection $h : [\ell] \rightarrow L$ where $L \subseteq S$ is any subset of cardinality ℓ . Then by Lemma 3.1, the complete Sperner family $\mathcal{S} = \{h(X) \cup (S - L) : X \in \mathcal{A}^\vee\}$ has linear chromatic number ℓ .

For the upper bound, note that Proposition 2.3 allows us to reduce our question to that of finding the size of a maximum antichain in a chain product $\tilde{C}_{m_1, \dots, m_n}$, where m_1, \dots, m_n are positive integers whose sum is m . Since these posets have the Sperner property by Proposition 2.2, the size of the largest antichain is equal to $|X_{\lfloor m/2 \rfloor}| = \{(x_1, \dots, x_n) : \sum_{i=1}^n x_i = \lfloor m/2 \rfloor, 0 \leq x_i \leq m_i\}$. Hence, the problem is reduced to finding the values of m_1, \dots, m_n which maximize this quantity. By Lemma 4.1, this is achieved by choosing each m_i to be either $\lfloor m/n \rfloor$ or $\lceil m/n \rceil$; clearly there is only one way of achieving this, up to permutation of the m_i . To show tightness of this bound, write $m = qn + r$ for $q, r \in \mathbb{Z}$ and $0 \leq r < n$, and partition S into n subsets S_1, \dots, S_n where $|S_i| = \lceil m/n \rceil$ for $1 \leq i \leq r$ and $|S_i| = \lfloor m/n \rfloor$ for $r + 1 \leq i \leq n$. Let $\mathcal{P} = \{S_1, \dots, S_n\}$ and fix any ranking R of \mathcal{P} . Then $\mathcal{C} = \{(x_1, \dots, x_n) \in \tilde{C}_{|S_1|, \dots, |S_n|} : x_1 + \dots + x_n = \lfloor m/2 \rfloor\}$ is an antichain in $\tilde{C}_{|S_1|, \dots, |S_n|}$ and $\mathcal{S} = \Phi(\mathcal{P}, R, \mathcal{C})$ is a Sperner family on S of size $s(m, n)$. \square

Note that when $n = m$, the upper bound of Theorem 4.2 is merely $\binom{m}{\lfloor m/2 \rfloor}$. This is not surprising in view of the fact that the Sperner family of maximum size in S , which consists of all subsets of size $\lfloor m/2 \rfloor$, has linear chromatic number m , since any pair of elements in S is separated by some pair of subsets in this family.

5 Sperner families of maximum linear chromatic number

Another question raised in [1] deals with the unimodality of the numbers $ls(m; n)$; in particular, does there exist $k \leq m$ such that $ls(m; 1) \leq \dots \leq ls(m; k) \geq ls(m; k+1) \geq \dots \geq ls(m; m)$? In this section, we show that in general, $ls(m; m)$ is far larger than $\sum_{n < m} ls(m; n)$; thus the only hope for unimodality of the numbers would lie in a proof that they are increasing as a function of n .

To this end, we need a result of Jeff Kahn. For a poset P , we denote by $a(P)$ the number of antichains in P . If P and Q are posets with the same underlying set, we say that Q is a *relaxation* of P if whenever y covers x in Q , y also covers x in P ; it is clear that $a(Q) \geq a(P)$.

Theorem 5.1. ([5, Theorem 1.3]) *Let P be a graded poset with levels P_1, \dots, P_k , with $|P_k| \leq M$. Suppose $d_+(x) \geq n$ for all $x \in P_1 \cup \dots \cup P_{k-1}$ and $d_-(x) \leq n$ for all $x \in P_2 \cup \dots \cup P_k$. Then $a(P) \leq (k2^n - (k-1))^{M/n}$.*

For our purposes, the weaker inequality $a(P) \leq k^{M/n} 2^M$ suffices.

Proposition 5.2. *There are at least $2^{\binom{m}{\lfloor m/2 \rfloor}} - m2^{\binom{m-1}{\lfloor m/2 \rfloor}}$ complete Sperner families on a set of cardinality m .*

Proof.

Suppose $|S| = m$, let $\mathcal{B} = \{B \subseteq S : |B| = \lfloor m/2 \rfloor\}$. Observe that each of the $2^{\mathcal{B}} = 2^{\binom{m}{\lfloor m/2 \rfloor}}$ subsets $\mathcal{S} \subseteq \mathcal{B}$ is a Sperner family. By Proposition 2.7 it suffices to enumerate the primitive families. If \mathcal{S} is imprimitive, it must be a subset of $\mathcal{B}_i = \{B \in \mathcal{B} : i \in B\}$ for some $i = 1, \dots, m$. Since $|\mathcal{B}_i| = \binom{m-1}{\lfloor m/2 \rfloor}$, it follows that there are at least $2^{\binom{m}{\lfloor m/2 \rfloor}} - m2^{\binom{m-1}{\lfloor m/2 \rfloor}}$ complete Sperner families on m . \square

Remark.

The result of Proposition 5.2 indicates that (as one might expect) most Sperner families are complete, at least when m is sufficiently large. Indeed, $\frac{\binom{m-1}{\lfloor m/2 \rfloor}}{\binom{m}{\lfloor m/2 \rfloor}} = \frac{m - \lfloor m/2 \rfloor}{m} \leq \left(\frac{1}{2} + \frac{1}{2m}\right) \frac{\binom{m}{\lfloor m/2 \rfloor}}{\binom{m}{\lfloor m/2 \rfloor}}$, so for large values of m , $m2^{\binom{m-1}{\lfloor m/2 \rfloor}} \leq 2^{\log_2 m + (\frac{1}{2} + \frac{1}{2m}) \binom{m}{\lfloor m/2 \rfloor}}$ is much smaller than $2^{\binom{m}{\lfloor m/2 \rfloor}}$.

To study linear Sperner families \mathcal{S} of chromatic number at most $m-1$ on a set S with m elements, first note that such families (being $m-1$ -linearly colorable), are

determined by a partition of S into $m-2$ singletons and one doubleton, an ordering of the two elements within that doubleton, and an antichain in the poset $P = \tilde{C}_{2, \underbrace{1, \dots, 1}_{m-2}}$.

Clearly there are $\frac{m(m-1)}{2}$ such doubletons and two orderings for each choice, so to obtain an upper bound on the number of such \mathcal{S} , it remains to bound the number of antichains in P . To this end, we use Theorem 5.1, following a construction similar to that found in [5, Prop. 5.1]. Note that P is a ranked poset with levels L_i , $i = 0, \dots, m$, where $\ell_i = |L_i| = \binom{m-2}{i} + \binom{m-2}{i-1} + \binom{m-2}{i-2} = \binom{m-1}{i} + \binom{m-2}{i-2}$. Since chain products are rank-symmetric by Proposition 2.2, the $m/2$ th is the largest level when m is even and the $\lfloor m/2 \rfloor$ and $\lceil m/2 \rceil$ levels are the largest levels when m is odd.

Viewing P as a directed graph whose arcs originate at vertices in L_i and terminate at vertices in L_{i+1} , observe that for $x \in L_i$, $i-1 \leq d_-(x) \leq i$ and $m-i-1 \leq d_+(x) \leq m-i$. For each $i > \lceil m/2 \rceil$ and each vertex x in level i with $d_-(x) > \lceil m/2 \rceil$ arcs, delete (arbitrarily) $d_-(x) - \lceil m/2 \rceil$ arcs ending at x , and define P' to be the resulting poset. Observe that P' is a relaxation of P .

First suppose m is even. We describe the construction of a poset Q of which P' is a subposet. For each $i > m/2$, add $\ell_{m/2} - \ell_i$ new vertices to each the i th level, and add new arcs originating at vertices in L_i terminating at these new vertices in such a way that for all x in the (new) i th level and y in the (new) $(i+1)$ st level, $d_+(x) = d_-(y) = m/2$. Denote by Q the resulting poset. Now Q satisfies the hypotheses of Theorem 5.1 with $k = m+1$, $n = m/2$, and

$$M = \ell_{m/2} = \binom{m-1}{m/2} + \binom{m-2}{m/2} = \left(\frac{1}{2} + \frac{1}{4}\left(1 - \frac{2}{m}\right)\left(1 - \frac{1}{m-1}\right)\right) \binom{m}{m/2},$$

so $a(P) \leq a(P') \leq a(Q) \leq 2^{\frac{2 \log_2(m+1)}{m} + (\frac{1}{2} + \frac{1}{4}(1 - \frac{2}{m})(1 - \frac{1}{m-1})) \binom{m}{m/2}}$.

If m is odd, the construction of Q is slightly more complicated. Direct computation shows that for a vertex x in level $\lfloor m/2 \rfloor$ (of P'), $d_+(x)$ is either $\lfloor m/2 \rfloor$ or $\lceil m/2 \rceil$; similarly, for a vertex y in level $\lceil m/2 \rceil$, $d_-(y)$ is either $\lfloor m/2 \rfloor$ or $\lceil m/2 \rceil$. Let $S = \{y \in L_{\lceil m/2 \rceil} : d_-(y) = \lfloor m/2 \rfloor\}$; note that S consists of all vertices of the form $(2, a_2, \dots, a_{m-1})$, where each a_i equals 0 or 1, so

$$|S| = \binom{m-2}{\lfloor m/2 \rfloor - 2} = \binom{m-2}{(m-3)/2} = \frac{1}{4} \left(1 + \frac{1}{m}\right) \binom{m}{\lfloor m/2 \rfloor}.$$

Add a set A of $h = \lceil |S| / \lfloor m/2 \rfloor \rceil$ new vertices to level $\lfloor m/2 \rfloor$ and, for each vertex in $y \in S$, add one new arc originating at some vertex of A and terminating at y . Observe

also that $\ell \leq \left(\frac{1}{2m} + \frac{1}{2m^2}\right) \binom{m}{\lfloor m/2 \rfloor}$. At this stage in the construction, $d_-(y) = \lceil m/2 \rceil$ for each y in level $\lceil m/2 \rceil$ and $d_+(x) \leq \lceil m/2 \rceil$ for each x in level $\lfloor m/2 \rfloor$. Now add a set B of h new vertices to level $\lceil m/2 \rceil$ and add new arcs originating at vertices in level $\lfloor m/2 \rfloor$ and terminating at vertices in B in such a way that for all x in the new i th level and y in the new $(i+1)$ st level, $d_+(x) = d_-(y) = \lceil m/2 \rceil$. Now for each $i > \lceil m/2 \rceil$, add $(\ell_{\lfloor m/2 \rfloor} + h) - \ell_i$ new vertices to each the i th level, and add new arcs originating at vertices in L_i terminating at these new vertices in such a way that for all x in the (new) i th level and y in the (new) $(i+1)$ st level, $d_+(x) = d_-(y) = \lceil m/2 \rceil$. Denote by Q the resulting poset. Now Q satisfies the hypotheses of Theorem 5.1 with $k = m+1$, $n = \lceil m/2 \rceil$, and $M = \ell_{\lfloor m/2 \rfloor} + h \leq \frac{3}{4} \left(1 + \frac{1}{m}\right) \binom{m}{\lfloor m/2 \rfloor} + \left(\frac{1}{2m} + \frac{1}{2m^2}\right) \binom{m}{\lfloor m/2 \rfloor} = \left(\frac{3}{4} + \frac{5}{4m} + \frac{1}{2m^2}\right) \binom{m}{\lfloor m/2 \rfloor}$, so $a(P) \leq a(P') \leq a(Q) \leq 2^{\frac{2 \log_2(m+1)}{m+1} + \left(\frac{3}{4} + \frac{5}{4m} + \frac{1}{2m^2}\right) \binom{m}{\lfloor m/2 \rfloor}}$.

Thus we have proven the following:

Proposition 5.3. *The number of Sperner families on a set m of linear chromatic number at most $m-1$ is bounded above by:*

$$2^{\log_2(m) + \log_2(m-1) + \frac{2 \log_2(m+1)}{m} + \left(\frac{3}{4} + \frac{5}{4m} + \frac{1}{2m^2}\right) \binom{m}{\lfloor m/2 \rfloor}}$$

In particular, for any $\varepsilon > 0$, the number of such families is at most $2^{\left(\frac{3}{4} + \varepsilon\right) \binom{m}{\lfloor m/2 \rfloor}}$ for sufficiently large m .

Combining this with the estimate of Proposition 5.2 and the following remark:

Corollary 5.4. *On a set of cardinality m , there are at least*

$$2^{\binom{m}{\lfloor m/2 \rfloor}} - 2^{\log_2 m + \left(\frac{1}{2} + \frac{1}{2m}\right) \binom{m}{\lfloor m/2 \rfloor}} - 2^{\log_2(m) + \log_2(m-1) + \frac{2 \log_2(m+1)}{m} + \left(\frac{3}{4} + \frac{5}{4m} + \frac{1}{2m^2}\right) \binom{m}{\lfloor m/2 \rfloor}}$$

Sperner families of linear chromatic number m . In particular, for any $\varepsilon > 0$, the number of such families is at least

$$2^{\binom{m}{\lfloor m/2 \rfloor}} - 2^{(3/4 + \varepsilon) \binom{m}{\lfloor m/2 \rfloor}} - 2^{(1/2 + \varepsilon) \binom{m}{\lfloor m/2 \rfloor}}$$

for sufficiently large m .

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